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Conformally Covariant Differential Operators: Properties and Applications

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Abstract

We discuss conformally covariant differential operators, which under local rescalings of the metric, $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$, transform according to $\delta_\sigma \Delta = r\Delta\sigma + (s-r)\sigma\Delta$ for some r if Δ is s th order. It is shown that the flat space restrictions of their associated Green functions have forms which are strongly constrained by flat space conformal invariance. The same applies to the variation of the Green functions with respect to the metric. The general results are illustrated by finding the flat space Green function and also its first variation for previously found second order conformal differential operators acting on k-forms in general dimensions. Furthermore we construct a new second order conformally covariant operator acting on rank four tensors with the symmetries of the Weyl tensor whose Green function is similarly discussed. We also consider fourth order operators, in particular a fourth order operator acting on scalars in arbitrary dimension, which has a Green function with the expected properties. The results obtained here for conformally covariant differential operators are generalisations of standard results for the two dimensional Laplacian on curved space and its associated Green function which is used in the Polyakov effective gravitational action. It is hoped that they may have similar applications in higher dimensions.

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1 Introduction

In discussions of the effects of quantum matter on gravity the effective action $W[g]$, for some fixed background metric $g_{\mu\nu}$, is of crucial interest. It is a scalar under diffeomorphisms, i.e. local coordinate reparametrisations when of course $g_{\mu\nu}$ transforms as a tensor field. The energy momentum tensor is defined by the response of the field theory to variations of the metric so that its expectation value for the background metric $g_{\mu\nu}$ is given in terms of the effective action by

$$\langle T_{\mu\nu}(x)\rangle = -\frac{2}{\sqrt{g(x)}}\frac{\delta}{\delta g^{\mu\nu}(x)}W[g]\,. \tag{1.1}$$

Here we consider quantum field theories where the trace of the energy momentum tensor has zero contributions from the quantum matter fields. Such field theories on curved space are expected to be also Weyl invariant, i.e. invariant under local rescalings of the metric. However in even dimensions there are local anomalies such that the effective action is not invariant under Weyl rescalings and the energy momentum tensor expectation value acquires an anomalous trace involving the curvature. In two dimensions this anomalous trace is simply given by

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = \frac{c}{24\pi}\,R\,, \tag{1.2}$$

with R the Ricci scalar and c the Virasoro central charge. In four dimensions the expression for the anomalous trace is more complicated. It contains at least two linearly independent dimension 4 scalars F, G formed from the metric so that

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = -\beta_a\,F - \beta_b\,G \tag{1.3}$$

where β_a and β_b are coefficients depending on the particular theory and

$$\begin{aligned} F \ &= \ R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - \frac{4}{d-2}\,R^{\alpha\beta}R_{\alpha\beta} + \frac{2}{(d-2)(d-1)}\,R^2 \ = \ C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}\,, && (1.4)\\ G \ &= \ R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - 4\,R^{\alpha\beta}R_{\alpha\beta} + R^2 \ = \ \tfrac{1}{4}\varepsilon^{\mu\nu\sigma\rho}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\sigma\rho}\,. && (1.5)\end{aligned}$$

$C_{\alpha\beta\gamma\delta}$ is the Weyl tensor which is given by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - 2\left(g_{\alpha[\gamma}K_{\delta]\beta} - g_{\beta[\gamma}K_{\delta]\alpha}\right)\,, \tag{1.6}$$

with

$$K_{\alpha\beta} = \frac{1}{d-2}\left(R_{\alpha\beta} - \frac{1}{2(d-1)}g_{\alpha\beta}R\right). \tag{1.7}$$

F and G are the only necessary gravitational contributions to the trace anomaly in general. Although a term proportional to R^2 might be expected in the trace (1.3), it must be absent here as a consequence of Wess-Zumino consistency conditions [1]. Moreover possible contributions to the trace anomaly proportional to $\nabla^2 R$ may be cancelled by adding a local term to the effective action.

In two dimensions there is a well-known unique expression for the effective action due to Polyakov [2], which is obtained by integrating the anomaly (1.2),

$$W^P[g] = -\frac{c}{96\pi}\iint \mathrm{d}^2x\mathrm{d}^2y\,\sqrt{g(x)}R(x)G(x,y)\sqrt{g(y)}R(y)\,, \tag{1.8}$$

with

$$-\sqrt{g(x)}\nabla_x^2 G(x,y) = \delta^2(x-y)\,. \tag{1.9}$$

The Polyakov action is manifestly a diffeomorphism invariant scalar. It has a non-local structure as it involves the Green function of a second order differential operator ∇^2 which transforms simply under local rescalings of the metric $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$,

$$\delta_\sigma(\sqrt{g}\nabla^2) = 0\,. \tag{1.10}$$

The Green function $G(x,y)$ associated to ∇^2 is therefore invariant under local rescalings of the metric,

$$\delta_\sigma G(x,y) = 0\,. \tag{1.11}$$

With $\delta_\sigma\sqrt{g}R = 2\sqrt{g}\nabla^2\sigma$ it is easy to check that the variation of the Polyakov action (1.8) gives the conformal anomaly (1.2).

In four dimensions it is also natural to consider possible non-local constructions for the effective action involving Green functions G_Δ associated with differential operators Δ with simple properties under Weyl rescalings and which reproduces the anomaly (1.3). To this end in this paper we analyse such differential operators, which we refer to here as conformally covariant differential operators, and derive some general properties of their associated Green functions. The simplest such operator in d dimensions is the second order operator

$$\Delta_2 = -\nabla^2 + \frac{d-2}{4(d-1)}R \tag{1.12}$$

acting on scalar fields, with the variation

$$\delta_\sigma\Delta_2 = \tfrac{1}{2}(d+2)\sigma\Delta_2 - \tfrac{1}{2}(d-2)\Delta_2\sigma\,. \tag{1.13}$$

Its associated Green function satisfies

$$\delta_\sigma G_2(x,y) = \tfrac{1}{2}(d-2)(\sigma(x)+\sigma(y))G_2(x,y)\,. \tag{1.14}$$

Clearly in two dimensions Δ reduces to $-\nabla^2$ and the Green function to $G(x,y)$ which featured in the above result for the Polyakov action. We investigate here generalisations to operators acting on various tensor fields in arbitrary dimensions d. The construction of conformally covariant differential operators, and also conformal invariants involving the Riemann curvature tensor, have been discussed extensively by mathematicians. In particular we may mention the results, in the general framework of differential geometry, found by Branson [3], Fefferman and Graham [4], Parker and Rosenberg [5], and by

Wünsch [6]. Here we use especially the second order differential operator acting on k-forms in d dimensions first obtained by Branson [3]. We also construct another conformally covariant second order differential operator which acts on tensor fields with the symmetries of the Weyl tensor defined in (1.6).

A crucial observation for our subsequent discussion is that theories defined on curved space which are invariant under diffeomorphisms and also Weyl rescalings of the metric are expected to be conformally invariant when reduced to flat space, when $g_{\mu\nu} = \delta_{\mu\nu}$. Conformal invariance is a strong symmetry constraint for field theories which allows for exact results for the two and three point correlation functions. Besides massless free field theories the physical relevance of conformal invariance is given by the fact that it should be realised for interacting quantum field theories at renormalisation group fixed points.

This property is illustrated by the Polyakov action since it is in agreement with conformal flat Euclidean space correlation functions [7]. We define the two point function on flat space to be

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(y)\rangle \;=\; 4\frac{\delta^2}{\delta g^{\mu\nu}(x)\delta g^{\sigma\rho}(y)}\, W^P[g]\Big|_{g=\delta}\,, \tag{1.15}$$

with a similar expression for the three point function. Although $W^P[g]$ itself is zero on flat space where the curvature vanishes, second or higher functional derivatives of $W^P[g]$ are non-zero even on flat space. Using $\delta_g(\sqrt{g}R) = -\sqrt{g}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\nabla^2)\delta g^{\mu\nu}$ and restricting to flat space, when $G(x,y)|_{g=\delta} = -\ln\mu^2(x-y)^2/4\pi$ with μ an arbitrary scale, we obtain

$$\langle T_{\mu\nu}(x) T_{\sigma\rho}(y)\rangle \;=\; -\frac{c}{48\pi^2} S^x{}_{\mu\nu} S^y{}_{\sigma\rho} \ln(x-y)^2\,, \quad S_{\mu\nu} = \partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2\,. \tag{1.16}$$

With conventional complex coordinates z on the plane, so that $x^2 = z\bar{z}$, and defining $T(z) = -2\pi T_{zz}(x)$, this gives rise to the standard two dimensional conformal field theory result

$$\langle T(z_1)T(z_2)\rangle = -\frac{c}{12}\,\partial_{z_1}{}^2\partial_{z_2}{}^2 \ln(x_1-x_2)^2 = \frac{\frac{1}{2}c}{(z_1-z_2)^4}\,. \tag{1.17}$$

Thus this calculation shows that the conformal energy momentum tensor two point function on flat space is completely determined by the conformal anomaly (1.2) on curved space. If we use for the variation of the Green function

$$\frac{\delta}{\delta g^{zz}(x_3)} G(x_1,x_2)\Big|_{g=\delta} = -\frac{1}{(4\pi)^2}\,\frac{1}{(z_1-z_3)(z_2-z_3)}\,, \tag{1.18}$$

then the standard result for the three point function at non coincident points,

$$\begin{aligned}\langle T(z_1)T(z_2)T(z_3)\rangle &= -\frac{c}{3}\Big\{\frac{1}{(z_1-z_3)^3(z_2-z_3)^3} + \frac{1}{(z_2-z_1)^3(z_3-z_1)^3} + \frac{1}{(z_1-z_2)^3(z_3-z_2)^3}\Big\} \\ &= \frac{c}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_1)^2}\,, \end{aligned}\tag{1.19}$$

may be similarly obtained by varying the Polyakov action (1.8) three times with respect to the metric [7].

In four dimensions there is no analogue of the Polyakov action at present which has all its properties, i.e. which just yields the conformal anomaly (1.3) under Weyl rescalings, and which also leads to conformally invariant correlation functions for the energy momentum tensor on flat space. An action constructed by Riegert [8], which involves the Green function of a conformally covariant fourth order operator acting on scalars, generates the conformal anomaly upon Weyl rescalings. However this does not lead to conformal correlation functions on flat space. As discussed in [7], this is due to the insufficiently rapid fall off at long-distances of the Riegert action, which is responsible for surface terms spoiling the conformal invariance Ward identities. This is related to the fact, pointed out by Deser [9], that the Green function for the fourth order differential operator which appears in the Riegert action has a double pole. A general construction for the four dimensional effective action to third order in the curvature, in terms of a basis for non-local invariants, has been constructed by Barvinsky et al. [10]. This is unfortunately very complicated and it is not at all clear which minimal linear combination of these terms is sufficient for an action with the required properties.

The organisation of this paper is as follows: We begin by discussing conformal differential operators on curved space and their associated Green functions in all generality in section 2. It is shown how the form of the variation of the Green function with respect to the metric is constrained by conformal invariance when reduced to flat space. In section 3 we apply our general results to the second order differential operator on k-forms constructed by Branson. In section 4 we discuss some previous results for scalars which transform homogeneously under local rescalings of the metric, i.e. infinitesimally $\delta_\sigma \mathcal{O} = \eta\sigma\mathcal{O}$ for dimension η. We rederive the expression for a scalar of dimension 6 in which was found by Fefferman and Graham [4] in addition to the two dimension 6 scalars which may be constructed trivially in terms of the Weyl tensor alone. With similar results we then construct a conformal second order differential operator acting on tensors with Weyl symmetry in section 5. We also discuss the Green function of this operator. In section 6 we consider fourth order operators, in particular an operator acting on scalars which reduces to the operator whose Green function is used in the Riegert action in four dimensions. Results compatible with conformal invariance are again obtained. Finally in section 7 we describe how the Riegert action is constructed but also that the fourth order operator has 4 as a critical dimension so that conformal invariant results are not in general obtained when reducing to flat space.

2 Conformally Covariant Differential Operators

We consider fields $\mathcal{O}^i(x)$ which are sections of tensor bundles over d-dimensional Riemannian space with metric $g^{\mu\nu}$. Under diffeomorphisms $\delta x^\mu = v^\mu(x)$, i.e. local coordinate reparametrisations, these fields transform as

$$\delta_v \mathcal{O}^i(x) = \mathcal{L}_{v(x)} \mathcal{O}^i(x) \,, \tag{2.1}$$

with $\mathcal{L}_v$ the appropriate Lie derivative. Acting on scalars φ and on the metric $g^{\mu\nu}$, $\mathcal{L}_v$ is given by

$$\mathcal{L}_v\varphi = v^\lambda\partial_\lambda\varphi\,, \quad \begin{aligned}\mathcal{L}_v g^{\mu\nu} &= v^\lambda\partial_\lambda g^{\mu\nu} - \partial_\lambda v^\mu g^{\lambda\nu} - \partial_\lambda v^\nu g^{\mu\lambda} \\ &= -\nabla^\mu v^\nu - \nabla^\nu v^\mu\,.\end{aligned} \tag{2.2}$$

It is also convenient to define the field $\bar{\mathcal{O}}_i(x)$ conjugate to $\mathcal{O}^i(x)$ such that $\bar{\mathcal{O}}_i\mathcal{O}^i$ is a scalar, $\delta_v(\bar{\mathcal{O}}_i\mathcal{O}^i) = v^\mu\partial_\mu(\bar{\mathcal{O}}_i\mathcal{O}^i)$. The transformation property is written as

$$\delta_v\bar{\mathcal{O}}_i(x) = \bar{\mathcal{L}}_{v(x)}\bar{\mathcal{O}}_i(x)\,. \tag{2.3}$$

Thus corresponding to (2.2)

$$\bar{\mathcal{L}}_v\varphi = \mathcal{L}_v\varphi\,, \quad \bar{\mathcal{L}}_v g_{\mu\nu} = v^\lambda\partial_\lambda g_{\mu\nu} + \partial_\mu v^\lambda g_{\lambda\nu} + \partial_\nu v^\lambda g_{\mu\lambda} \tag{2.4}$$

when acting on scalars or on the metric. Furthermore under local Weyl rescalings of the metric,

$$\delta_\sigma g^{\mu\nu}(x) = 2\,\sigma(x) g^{\mu\nu}(x)\,, \tag{2.5}$$

we require

$$\delta_\sigma\mathcal{O}^i(x) = -r\,\sigma(x)\mathcal{O}^i(x)\,, \quad \delta_\sigma\bar{\mathcal{O}}_i(x) = -\bar{r}\,\sigma(x)\bar{\mathcal{O}}_i(x)\,, \tag{2.6}$$

with some real numbers $r, \bar{r}$.

Now let Δ be an elliptic differential operator of order s. Under diffeomorphisms and Weyl rescalings we then assume

$$\delta_v(\Delta\mathcal{O})^i(x) = \mathcal{L}_{v(x)}(\Delta\mathcal{O})^i(x) \tag{2.7}$$

$$\delta_\sigma(\Delta\mathcal{O})^i(x) = (s-r)\sigma(x)(\Delta\mathcal{O})^i(x)\,. \tag{2.8}$$

(2.6) and (2.8) imply

$$\delta_\sigma\Delta = r\Delta\sigma + (s-r)\sigma\Delta\,. \tag{2.9}$$

Since $\delta_v\sqrt{g} = \sqrt{g}\nabla\cdot v$ the expression

$$S_\mathcal{O} = \int \mathrm{d}^d x\,\sqrt{g}\,\bar{\mathcal{O}}_i(\Delta\mathcal{O})^i \tag{2.10}$$

is then an invariant scalar, $\delta_v S_\mathcal{O} = 0$, and moreover this is also Weyl invariant, $\delta_\sigma S_\mathcal{O} = 0$, if

$$\bar{r} + r - s = -d\,. \tag{2.11}$$

The equation

$$\sqrt{g(x)}(\Delta_x G_\Delta)^i{}_j(x,y) = \delta^i{}_j\delta^d(x-y) \tag{2.12}$$

defines the Green function $G_\Delta{}^i{}_j(x,y)$ of the operator Δ. $\delta^i{}_j$ is the identity for the space of tensors under consideration. Under diffeomorphisms this Green function transforms as

$$\delta_v G_\Delta{}^i{}_j(x,y) = \mathcal{L}_{v(x)}G_\Delta{}^i{}_j(x,y) + \bar{\mathcal{L}}_{v(y)}G_\Delta{}^i{}_j(x,y)\,. \tag{2.13}$$

The right hand side of (2.12) is invariant under Weyl rescalings, which implies together with (2.9) that

$$\delta_\sigma G_\Delta{}^i{}_j(x,y) = (d-r)\sigma(x) G_\Delta{}^i{}_j(x,y) - (s-r)\sigma(y) G_\Delta{}^i{}_j(x,y)\,. \tag{2.14}$$

Thus the conditions for Weyl and diffeomorphism invariance of the Green function implied by (2.12), (2.13) and (2.14) are

$$\begin{aligned}\Big((d-r)\sigma(x) - (s-r)\sigma(y)\Big) G_\Delta{}^i{}_j(x,y) \qquad & \\ + 2\int \mathrm{d}^d z\, \sigma(z) g^{\alpha\beta}(z) \frac{\delta}{\delta g^{\alpha\beta}(z)} G_\Delta{}^i{}_j(x,y) &= 0\,, \qquad (2.15)\\ (\mathcal{L}_{v(x)} + \bar{\mathcal{L}}_{v(y)}) G_\Delta{}^i{}_j(x,y) + \int \mathrm{d}^d z\, \mathcal{L}_{v(z)} g^{\alpha\beta}(z) \frac{\delta}{\delta g^{\alpha\beta}(z)} G_\Delta{}^i{}_j(x,y) &= 0\,. \qquad (2.16)\end{aligned}$$

The sum of these two equations gives a non-trivial condition on G_Δ for a fixed metric if we restrict v, $\sigma = \sigma_v$ by

$$\mathcal{L}_v g^{\alpha\beta} + 2\sigma_v g^{\alpha\beta} = 0\,, \quad \sigma_v = \nabla{\cdot}v/d\,. \tag{2.17}$$

With this condition, (2.15) and (2.16) yield

$$\Big(\mathcal{L}_{v(x)} + \bar{\mathcal{L}}_{v(y)} + (d-r)\sigma_v(x) - (s-r)\sigma_v(y)\Big) G_\Delta{}^i{}_j(x,y) = 0\,, \tag{2.18}$$

although v, σ_v must be constrained so that surface terms can be neglected. Restricting to flat Euclidean space, (2.17) gives the conformal Killing equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma_v \delta_{\mu\nu}\,, \tag{2.19}$$

which ensures that $\delta x_\mu = v_\mu(x)$ is a conformal transformation, and any Green function satisfying (2.18) is conformally covariant on flat space.

On flat space by translational invariance the Green function $G_\Delta(x,y)$ depends only on $x-y$, so that denoting the flat space Green function by $\mathring{G}_\Delta$ we have

$$\mathring{G}_\Delta{}^i{}_j(x-y) = G_\Delta{}^i{}_j(x,y)\Big|_{g=\delta}\,. \tag{2.20}$$

However applying the results for conformal two point functions developed in [11] gives an explicit form for $\mathring{G}_\Delta$ for conformal differential operators,

$$\mathring{G}_\Delta{}^i{}_j(x) = C_\Delta\, \frac{D^i{}_j\big(I(x)\big)}{(x^2)^{\frac{1}{2}(d-s)}}\,, \tag{2.21}$$

where $D^i{}_j(I(x))$ is here the appropriate representation of the inversion tensor acting on the fields $\mathcal{O}^i$ and C_Δ is some constant coefficient depending on d and the particular tensor representation. Inversions are conformal coordinate transformations for which

$$x'_\mu = \frac{x_\mu}{x^2}\,, \tag{2.22}$$

and the fundamental representation of the inversion tensor is given by

$$x'_\mu = x^2 I_{\mu\nu}(x) x_\nu \,, \quad I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \,, \ \ \det I = -1 \,. \tag{2.23}$$

The inversion is a conformal transformation not connected to the identity. Its significance for the two point functions is due to the fact that it plays the role of parallel transport for conformal transformations.

The variation of the Green function G_Δ with respect to a change in the metric is given by

$$\delta_g {G_\Delta}^i{}_j(x,y) = -\int \mathrm{d}^d z \, {G_\Delta}^i{}_k(x,z) \Big(\delta_g(\sqrt{g}\Delta_z) G_\Delta\Big)^k{}_j(z,y) \,. \tag{2.24}$$

If we thereby define

$$G'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) = \frac{\delta}{\delta g^{\alpha\beta}(z)} {G_\Delta}^i{}_j(x,y) \,, \tag{2.25}$$

then scale and diffeomorphism invariance imposed by virtue of (2.15) and (2.16) imply, for $z \neq x, y$, that

$$g^{\alpha\beta}(z) G'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) \ = \ 0 \,, \tag{2.26}$$

$$\nabla_z^\alpha G'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) \ = \ 0 \,. \tag{2.27}$$

Restricting to flat space,

$$\mathring{G}'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) = G'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z)\Big|_{g=\delta} \,, \tag{2.28}$$

which depends only on $x-z, y-z$, is also strongly constrained by conformal invariance. It may be expressed in the same form as was obtained for conformal three point functions in [11, 7], giving

$$\mathring{G}'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) = -\mathring{G}_\Delta{}^i{}_{i'}(x-z) \mathring{G}_\Delta{}^{j'}{}_j(z-y) P^{i'}{}_{j',\alpha\beta}(Z) \,. \tag{2.29}$$

P is a tensor symmetric and traceless in $(\alpha\beta)$ which by conformal invariance depends only on the conformal vector Z_μ. This transforms as a vector at z and is defined by

$$Z_\mu = \frac{(x-z)_\mu}{(x-z)^2} - \frac{(y-z)_\mu}{(y-z)^2} \,, \qquad Z^2 = \frac{(x-y)^2}{(x-z)^2(y-z)^2} \,. \tag{2.30}$$

The tensor P satisfies the homogeneity property

$$P^i{}_{j,\alpha\beta}(\lambda Z) = \lambda^s P^i{}_{j,\alpha\beta}(Z) \,. \tag{2.31}$$

Note that $P^i{}_{j,\alpha\beta}(Z)$ has the crucial property that it does not contain any factor $(Z^2)^{-n}$, $n = 1, 2, \ldots$, since (2.24) implies that the only singular contributions for two of the three points coincident involve $(x-z)$, $(y-z)$, but not $(x-y)$.

By using results from [7] the conservation equation (2.27) can be simplified so as to constrain $P^i{}_{j,\alpha\beta}$ alone. Instead of (2.29) we may alternatively write using results from [11, 7]

$$\mathring{G}'_\Delta{}^i{}_{j,\alpha\beta}(x,y;z) = -C_\Delta{}^2 \, \frac{I_{\alpha\epsilon}(z-x) I_{\beta\eta}(z-x) D^{j'}{}_j\Big(I(x-y)\Big)}{\Big((z-x)^2\Big)^d \Big((x-y)^2\Big)^{\frac{1}{2}(d-s)}} \tilde{P}^i{}_{j',\epsilon\eta}(X) \, . \tag{2.32}$$

where

$$\tilde{P}^i{}_{j,\alpha\beta}(X) = \frac{1}{(X^2)^{\frac{1}{2}(d+s)}} P^i{}_{j',\alpha\beta}(X) D^{j'}{}_j\Big(I(X)\Big) \, , \tag{2.33}$$

and

$$X_\mu = \frac{(y-x)_\mu}{(y-x)^2} - \frac{(z-x)_\mu}{(z-x)^2} \, . \tag{2.34}$$

Then on flat space (2.27) is equivalent to

$$\partial_\alpha \tilde{P}^i{}_{j,\alpha\beta}(X) = 0 \, . \tag{2.35}$$

3 Second Order Conformal Operator on k-Forms

As an example for the operator Δ introduced in the previous section, we now discuss the the conformal second order operator acting on k-forms in d dimensions which was constructed by Branson [3]. We begin with some definitions.

If $\mathcal{A} = (1/k!)\mathcal{A}_{\mu_1 \cdots \mu_k} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}$ is a k-form, the exterior derivative d acting on k-forms and its adjoint δ, satisfying $\mathrm{d}^2 = \delta^2 = 0$, are defined by

$$\begin{aligned}
(\mathrm{d}\mathcal{A})_{\mu_1 \cdots \mu_{k+1}} &= \sum_{j=1}^{k+1} (-1)^{j-1} \partial_{\mu_j} \mathcal{A}_{\mu_1 \cdots \hat{\mu}_j \cdots \mu_{k+1}} = (k+1) \partial_{[\mu_1} \mathcal{A}_{\mu_2 \cdots \mu_{k+1}]} \, , & (3.1) \\
(\delta\mathcal{A})_{\mu_1 \cdots \mu_{k-1}} &= -\nabla^\lambda \mathcal{A}_{\lambda \mu_1 \cdots \mu_{k-1}} \\
&= -\frac{1}{\sqrt{g}} \, g_{\mu_1\nu_1} \cdots g_{\mu_{k-1}\nu_{k-1}} \partial_\lambda \Big(\sqrt{g} g^{\lambda\tau} g^{\nu_1\rho_1} \cdots g^{\nu_{k-1}\rho_{k-1}} \mathcal{A}_{\tau\rho_1 \cdots \rho_{k-1}} \Big) \, , & (3.2)
\end{aligned}$$

where the hat $\hat{\mu}_j$ indicates that the corresponding index is to be omitted. Following Branson we also define, with $R_{\mu\nu}$ and R the Ricci tensor and scalar curvature,

$$\begin{aligned}
J &\equiv \frac{1}{2(d-1)} R \, , & (3.3) \\
K_{\mu\nu} &\equiv \frac{1}{(d-2)} \left(R_{\mu\nu} - J g_{\mu\nu} \right) \, , & (3.4)
\end{aligned}$$

so clearly $K_{\mu\nu}$ is the same tensor as used for the definition of the Weyl tensor in (1.6). These transform under local Weyl rescalings as

$$\delta_\sigma J = 2\sigma J + \nabla^2 \sigma \, , \quad \delta_\sigma K_{\mu\nu} = \nabla_\mu \nabla_\nu \sigma \, . \tag{3.5}$$

To construct a second order conformal differential operator acting on k-forms we first consider the variation of δd and $d\delta$ under $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$, $\delta_\sigma \sqrt{g} = -d\sigma\sqrt{g}$ using the explicit dependence on the metric exhibited in (3.2) to give, for $\gamma \equiv \frac{1}{2}(d-2k)$,

$$\begin{aligned}
\delta_\sigma(\delta \mathrm{d}\mathcal{A})_{\mu_1\cdots\mu_k} &= 2\gamma\sigma(\delta\mathrm{d}\mathcal{A})_{\mu_1\cdots\mu_k} \\
&\quad + 2(\gamma-1)(k+1)\nabla^\lambda\left(\sigma\nabla_{[\lambda}\mathcal{A}_{\mu_1\cdots\mu_k]}\right) \\
\delta_\sigma(\mathrm{d}\delta\mathcal{A})_{\mu_1\cdots\mu_k} &= -2\gamma(\mathrm{d}\delta(\sigma\mathcal{A}))_{\mu_1\cdots\mu_k} \\
&\quad - 2(\gamma+1)k\nabla_{[\mu_1}\left(\sigma\nabla^\lambda\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]}\right) \ .
\end{aligned} \tag{3.6}$$

Using

$$(k+1)\nabla^\lambda(\sigma\nabla_{[\lambda}\mathcal{A}_{\mu_1\cdots\mu_k]}) = \nabla^\lambda(\sigma\nabla_\lambda\mathcal{A}_{\mu_1\cdots\mu_k}) - k\nabla^\lambda(\sigma\nabla_{[\mu_1}\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]}) \tag{3.7}$$

we find

$$\begin{aligned}
&\delta_\sigma\Big((\gamma+1)(\delta\mathrm{d}\mathcal{A})_{\mu_1\cdots\mu_k} + (\gamma-1)(\mathrm{d}\delta\mathcal{A})_{\mu_1\cdots\mu_k})\Big) \\
&\quad = 2\gamma(\gamma+1)\sigma(\delta\mathrm{d}\mathcal{A})_{\mu_1\cdots\mu_k} + 2\gamma(\gamma-1)(\mathrm{d}\delta(\sigma\mathcal{A}))_{\mu_1\cdots\mu_k} \\
&\qquad - 2(\gamma-1)(\gamma+1)k\Big[\nabla^\lambda(\sigma\nabla_{[\mu_1}\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]}) + \nabla_{[\mu_1}(\sigma\nabla^\lambda\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]})\Big] \\
&\qquad + 2(\gamma-1)(\gamma+1)\nabla^\lambda(\sigma\nabla_\lambda\mathcal{A}_{\mu_1\cdots\mu_k}) \, .
\end{aligned} \tag{3.8}$$

Single derivatives acting on σ are then eliminated by virtue of the identity

$$\begin{aligned}
\nabla^\lambda\sigma\nabla_\mu + \nabla_\mu\sigma\nabla^\lambda &= \tfrac{1}{2}(\nabla^\lambda\nabla_\mu + \nabla_\mu\nabla^\lambda)\sigma \\
&\quad - \tfrac{1}{2}\sigma(\nabla^\lambda\nabla_\mu + \nabla_\mu\nabla^\lambda) - (\nabla_\mu\nabla^\lambda\sigma) \ ,
\end{aligned} \tag{3.9}$$

so that we obtain for the the conformal variation of the operator

$$\begin{aligned}
\mathcal{D}^{(k)} &\equiv (\gamma+1)\delta\mathrm{d} + (\gamma-1)\mathrm{d}\delta \\
&= (\gamma+1)(\delta\mathrm{d} + \mathrm{d}\delta) - 2\mathrm{d}\delta \, ,
\end{aligned} \tag{3.10}$$

from (3.8) the result

$$\begin{aligned}
\delta_\sigma(\mathcal{D}^{(k)}\mathcal{A})_{\mu_1\cdots\mu_k} &= (\gamma+1)\sigma(\mathcal{D}^{(k)}\mathcal{A})_{\mu_1\cdots\mu_k} - (\gamma-1)(\mathcal{D}^{(k)}\sigma\mathcal{A})_{\mu_1\cdots\mu_k} \\
&\quad + 2(\gamma-1)(\gamma+1)k(\nabla_{[\mu_1}\nabla^\lambda\sigma)\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]} \\
&\quad - (\gamma-1)(\gamma+1)\nabla^2\sigma\,\mathcal{A}_{\mu_1\cdots\mu_k} \ .
\end{aligned} \tag{3.11}$$

The last two terms involving second derivatives acting on σ can be cancelled by terms linear in J and $K_\mu{}^\nu$, using the results for their variation exhibited in (3.5). The resulting conformally covariant differential operator

$$\Delta^{(k)} = \mathcal{D}^{(k)} + (\gamma+1)(\gamma-1)\,(J - 2k\,K) \tag{3.12}$$

where

$$(K\mathcal{A})_{\mu_1\cdots\mu_k} \equiv K_{[\mu_1}{}^\nu\mathcal{A}_{|\nu|\mu_2\cdots\mu_k]} \, . \tag{3.13}$$

is identical with the formula for $\Delta^{(k)}$ constructed by Branson. From the above calculations this has the conformal variation as in (2.9) for $s=2$,

$$\delta_\sigma \Delta^{(k)} = (\gamma+1)\sigma\Delta^{(k)} - (\gamma-1)\Delta^{(k)}\sigma \,. \tag{3.14}$$

On 0-forms, or scalar fields, $\delta \to 0$ and therefore

$$\Delta^{(0)} = \tfrac{1}{2}(d+2)\left(\delta \mathrm{d} + \tfrac{1}{2}(d-2)J\right) = \tfrac{1}{2}(d+2)\Delta_2 \tag{3.15}$$

the operator reduces to the well known conformal differential operator defined in (1.12).

An alternative expression of the existence of the conformal differential operator acting on k-forms $\Delta^{(k)}$ given by (3.12) may be found in terms of the d-dimensional action for the k-form $\mathcal{A}_{\mu_1\dots\mu_k}$ given by

$$\begin{aligned} S^{(k)}(g,\mathcal{A}) \;=\; \tfrac{1}{2}\int \mathrm{d}^d x \, \sqrt{g}\, \Big[&(\gamma+1)\,(\mathrm{d}\mathcal{A}){\cdot}(\mathrm{d}\mathcal{A}) + (\gamma-1)\,(\delta\mathcal{A}){\cdot}(\delta\mathcal{A}) \\ &+ (\gamma+1)(\gamma-1)(J\,\mathcal{A}{\cdot}\mathcal{A} - 2k\,\mathcal{A}{\cdot}(K\mathcal{A}))\Big] \,, \end{aligned} \tag{3.16}$$

where for X, Y k-forms we define,

$$X{\cdot}Y = \frac{1}{k!}\, X^{\mu_1\dots\mu_k} Y_{\mu_1\dots\mu_k} \,. \tag{3.17}$$

The result (3.14) is then equivalent to

$$\delta_\sigma S^{(k)}(g,\mathcal{A}) = 0 \quad \text{if} \quad \delta_\sigma \mathcal{A} = (\gamma-1)\,\sigma\mathcal{A} \,. \tag{3.18}$$

It is convenient to define

$$2\frac{\delta}{\delta g^{\alpha\beta}} S^{(k)}(g,\mathcal{A}) = \sqrt{g}\, T^{(k)}{}_{\alpha\beta} \,, \qquad \mathring{T}^{(k)}{}_{\alpha\beta} = T^{(k)}{}_{\alpha\beta}\Big|_{g=\delta} \,. \tag{3.19}$$

The calculation of the flat space expression for $\mathring{T}^{(k)}{}_{\alpha\beta}$ is straightforward given the explicit form of d in (3.1), which is metric independent, and δ in (3.2) but the detailed formula is lengthy and so it is relegated to appendix A.4. From this we may find

$$\begin{aligned} \mathring{T}^{(k)}{}_{\alpha\alpha} &= -(\gamma-1)\,\mathcal{A}{\cdot}(\mathring{\mathcal{D}}^{(k)}\mathcal{A}) \,, \\ \partial_\alpha \mathring{T}^{(k)}{}_{\alpha\beta} &= -\tfrac{1}{k!}\,(\mathrm{d}\mathcal{A})_{\beta\mu_1\dots\mu_k}\,(\mathring{\mathcal{D}}^{(k)}\mathcal{A})_{\mu_1\dots\mu_k} - \tfrac{1}{(k-1)!}\,\mathcal{A}_{\beta\mu_1\dots\mu_{k-1}}\,(\delta\mathring{\mathcal{D}}^{(k)}\mathcal{A})_{\mu_1\dots\mu_{k-1}} \,, \end{aligned} \tag{3.20}$$

with $\mathring{\mathcal{D}}^{(k)}$ the flat space restriction of $\mathcal{D}^{(k)}$, as in (3.10), and hence of the conformal differential operator $\Delta^{(k)}$.

According to the definition (2.12), the Green function of the operator $\Delta^{(k)}$ is defined by

$$\sqrt{g(x)}\Big(\Delta_x^{(k)} G^{(k)}\Big)_{\mu_1\cdots\mu_k,}{}^{\nu_1\cdots\nu_k}(x,y) = \mathcal{E}^{A}{}_{\mu_1\cdots\mu_k,}{}^{\nu_1\cdots\nu_k}\,\delta^d(x-y) \,, \tag{3.21}$$

for $\mathcal{E}^A$ the projector onto totally antisymmetric k-index tensors,

$$\mathcal{E}^A{}_{\mu_1\cdots\mu_k,}{}^{\nu_1\cdots\nu_k} = \delta_{[\mu_1}{}^{\nu_1}\dots\delta_{\mu_k]}{}^{\nu_k} \,. \tag{3.22}$$

In appendix A.2 we calculate this Green function for general d and k on flat space when $g_{\mu\nu} = \delta_{\mu\nu}$ and we may identify up and down indices. The result is

$$\mathring{G}^{(k)}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = T_d\,\mathcal{I}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\frac{1}{x^{d-2}} \,, \tag{3.23}$$

where

$$T_d = \frac{1}{S_d}\,\frac{1}{2(\gamma-1)(\gamma+1)} \,, \qquad S_d = \frac{2\pi^{d/2}}{\Gamma(\frac{1}{2}d)} \tag{3.24}$$

and $\mathcal{I}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)$ is the appropriate k-form representation for inversions and is given by

$$\mathcal{I}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = \mathcal{E}^A{}_{\mu_1\cdots\mu_k,\sigma_1\cdots\sigma_k} I_{\sigma_1\nu_1}(x)\cdots I_{\sigma_k\nu_k}(x) \,. \tag{3.25}$$

Note that this Green function does not exist if $\gamma = \pm 1$. If $\gamma = 1$, or $k = \frac{1}{2}(d-2)$, then $\Delta^{(k)} = 2\delta\mathrm{d}$ and there are solutions satisfying $\Delta^{(k)}\psi = 0$ for $\psi = \mathrm{d}\phi$. When $\gamma = -1$, or $k = \frac{1}{2}(d+2)$, $\Delta^{(k)} = -2\mathrm{d}\delta$ so the operators are not invertible. The result for the Green function (3.23) is exactly of the form expected from the general results for conformal two point functions discussed in section 2 as given in (2.21).

The conformal variation of the operator $\Delta^{(k)}$ defined in (3.12) as given by (3.14) remains unchanged if we add a term involving the Weyl tensor

$$(\tilde{\Delta}^{(k)}\mathcal{A})_{\mu_1\cdots\mu_k} = (\Delta^{(k)}\mathcal{A})_{\mu_1\cdots\mu_k} + t\,C_{[\mu_1\mu_2}{}^{\sigma\rho}\mathcal{A}_{|\sigma\rho|\mu_3\cdots\mu_k]} \tag{3.26}$$

for any real t since $\delta_\sigma C_{\mu\nu}{}^{\sigma\rho} = 2\sigma\, C_{\mu\nu}{}^{\sigma\rho}$.

We may now calculate the variation of the Green function $\tilde{G}^{(k)}$ for this operator with respect to the metric using the formula (2.24). The variation of the differential operator $\tilde{\Delta}^{(k)}$ may be obtained by varying the expressions for $(\mathrm{d}\delta\mathcal{A})_{\mu_1\cdots\mu_k}$ and $(\delta\mathrm{d}\mathcal{A})_{\mu_1\cdots\mu_k}$ using (3.1,3.2), which give the explicit dependence on the metric, and also calculating directly the variation of $J, K_{\mu\nu}$ defined in (3.3,3.4), but is equivalently given by using the expressions obtained from (3.19). Assuming the result (2.29) simplifies the calculation considerably, since it is sufficient to only determine the most singular terms as $z \to y$ which arise from terms with two derivatives acting on $G^{(k)}{}_{\rho_1\cdots\rho_k,\nu_1\cdots\nu_k}(z,y)$, and we then find, with definitions analogous to (2.25,2.28),

$$\begin{aligned}\mathring{\tilde{G}}^{(k)\prime}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k,\alpha\beta}(x,y;z) = {}& -\mathring{G}^{(k)}{}_{\mu_1\cdots\mu_k,\sigma_1\cdots\sigma_k}(x-z)\mathring{G}^{(k)}{}_{\rho_1\cdots\rho_k,\nu_1\cdots\nu_k}(z-y)\,P_{\sigma_1\cdots\sigma_k,\rho_1\cdots\rho_k,\alpha\beta}(Z)\\ & - 2t\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\kappa\lambda\varepsilon\eta}\partial^z{}_\gamma\partial^z{}_\delta\left(\mathring{G}^{(k)}{}_{\mu_1\cdots\mu_k,\kappa\lambda\sigma_3\cdots\sigma_k}(x-z)\mathring{G}^{(k)}{}_{\varepsilon\eta\sigma_3\cdots\sigma_k,\nu_1\cdots\nu_k}(z-y)\right) \,,\end{aligned} \tag{3.27}$$

where $\mathcal{E}^C$ is a projection operator onto tensors with Weyl symmetry defined in appendix A.1 and $\mathring{G}^{(k)}$ is as in (3.23). The second line corresponds to the additional term in (3.26).

In the first line from direct calculation we find

$$\begin{aligned}
& P_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k,\alpha\beta}(Z) \\
& \quad = d\frac{(\gamma+1)(\gamma-1)}{(d-1)(d-2)}\Big[\, 2k(kd-d+k)\,\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\varepsilon\rho_2\cdots\rho_k}\mathcal{E}^A{}_{\eta\rho_2\cdots\rho_k,\nu_1\cdots\nu_k}\mathcal{E}^T{}_{\varepsilon\eta,\alpha\beta}Z^2 \\
& \qquad + 4k(k-1)(d+1)\gamma\mathcal{E}^A_{\mu_1\cdots\mu_k,\kappa\varepsilon\rho_3\cdots\rho_k}\mathcal{E}^A_{\lambda\eta\rho_3\cdots\rho_k,\nu_1\cdots\nu_k}\mathcal{E}^T_{\kappa\lambda,\alpha\beta}Z_\varepsilon Z_\eta \\
& \qquad + k(d^2-2kd-d-2k+2)\,\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\varepsilon\rho_2\cdots\rho_k}\mathcal{E}^A{}_{\eta\rho_2\cdots\rho_k,\nu_1\cdots\nu_k} \\
& \qquad\qquad \times(\mathcal{E}^T{}_{\varepsilon\kappa,\alpha\beta}Z_\eta+\mathcal{E}^T{}_{\eta\kappa,\alpha\beta}Z_\varepsilon)Z_\kappa \\
& \qquad - \tfrac{1}{2}(d^2-2kd-4d+4)\,\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(Z_\alpha Z_\beta-\tfrac{1}{d}\delta_{\alpha\beta}Z^2)\Big]\,,
\end{aligned} \tag{3.28}$$

with $\mathcal{E}^T{}_{\varepsilon\eta,\alpha\beta}=\frac{1}{2}(\delta_{\varepsilon\alpha}\delta_{\eta\beta}+\delta_{\varepsilon\beta}\delta_{\eta\alpha})-\frac{1}{d}\delta_{\varepsilon\eta}\delta_{\alpha\beta}$ the projector onto symmetric traceless tensors. The tracelessness equation (2.26) and conservation equation (2.27) follow directly for flat space at non coincident points from the result (3.20). We have verified that this result is in accord with the conservation equation (2.35) with $P\to\tilde P$ defined as in (2.33).

For the case $d=4$ and $k=2$ giving $\gamma=0$ the Green function and its variation have been used in [7] to construct an anomaly-free contribution to the effective action involving the field strength tensor of a background gauge field. For the scalar operator Δ_2 defined in (1.12) then, making use of (3.15), we have as a special case of the above

$$\mathring{G}_2(x) = \frac{1}{(d-2)S_d}\,\frac{1}{x^{d-2}}\,, \tag{3.29}$$

$$\mathring{G}'_{2,\alpha\beta}(x,y;z) = \frac{d(d-2)^2}{4(d-1)}\,\mathring{G}_2(x-z)\mathring{G}_2(y-z)\Big(Z_\alpha Z_\beta-\frac{1}{d}\delta_{\alpha\beta}Z^2\Big)\,. \tag{3.30}$$

In this case the conservation equation (2.35) simply reduces to

$$\partial_\alpha\Big(\frac{1}{(X^2)^{\frac{1}{2}d+2}}\Big(X_\alpha X_\beta-\frac{1}{d}\delta_{\alpha\beta}X^2\Big)\Big)=0\,. \tag{3.31}$$

4 Conformal Invariants

Before constructing a further conformal differential operator it is convenient to consider possible scalar fields constructed in terms of the metric which transform homogeneously under local Weyl rescalings of the metric. It is trivial to construct such scalars in terms of the Weyl tensor defined in (1.6), since for $\delta_\sigma g^{\mu\nu}=2\sigma g^{\mu\nu}$ this transforms as

$$\delta_\sigma C_{\mu\sigma\rho\nu} = -2\sigma\, C_{\mu\sigma\rho\nu}\,. \tag{4.1}$$

Thus F defined in (1.4) transforms as $\delta_\sigma F = 4\sigma F$. Using the symmetry and trace conditions

$$C_{\mu\alpha\beta\nu}=C_{[\mu\alpha][\beta\nu]}\,,\quad C_{\mu[\alpha\beta\nu]}=0\,,\quad g^{\mu\nu}C_{\mu\alpha\beta\nu}=0\,, \tag{4.2}$$

there are two such scalars cubic in the Weyl tensor,

$$\begin{aligned}\Omega_1 &= C^{\mu\sigma}{}_{\rho\nu}C^{\rho\nu}{}_{\alpha\beta}C^{\alpha\beta}{}_{\mu\sigma}\,, &(4.3)\\ \Omega_2 &= C^{\mu\sigma\rho\nu}C_{\nu\beta\sigma\alpha}C^{\alpha}{}_{\rho}{}^{\beta}{}_{\mu}\,, &(4.4)\end{aligned}$$

which, from (4.1), satisfy $\delta_\sigma\Omega_1 = 6\sigma\Omega_1$, $\delta_\sigma\Omega_2 = 6\sigma\Omega_2$. For general d these are independent but when $d = 4$, $\Omega_1 = 4\Omega_2$ since

$$5\,C^{\mu\sigma}{}_{[\rho\nu}C^{\rho\nu}{}_{\alpha\beta}C^{\alpha\beta}{}_{\mu]\sigma} = \Omega_1 - 4\Omega_2\,. \tag{4.5}$$

As shown by Fefferman and Graham [4] there is an additional scalar H satisfying

$$\delta_\sigma H = 6\sigma H\,. \tag{4.6}$$

We here verify the existence of a scalar satisfying (4.6) since in the physics literature results have been given only for Weyl rescaling invariant integrals over all space in six dimensions [2]. For this purpose it is useful to define the Cotton tensor

$$\tilde{C}_{\beta\gamma\delta} = \nabla_\delta K_{\beta\gamma} - \nabla_\gamma K_{\beta\delta}\,, \tag{4.7}$$

with $K_{\beta\gamma}$ as in (3.4). It satisfies the identities

$$\tilde{C}_{\beta\gamma\delta} = -\tilde{C}_{\beta\delta\gamma} \quad \text{and} \qquad \tilde{C}_{\beta\gamma\delta} + \tilde{C}_{\gamma\delta\beta} + \tilde{C}_{\delta\beta\gamma} = 0\,, \tag{4.8}$$

and by virtue of the standard Bianchi identities

$$C_{\alpha\beta[\gamma\delta;\varepsilon]} = \tilde{C}_{\alpha[\gamma\delta}g_{\varepsilon]\beta} - \tilde{C}_{\beta[\gamma\delta}g_{\varepsilon]\alpha}\,. \tag{4.9}$$

From this result we may easily obtain

$$\nabla^\alpha C_{\alpha\beta\gamma\delta} = -(d-3)\tilde{C}_{\beta\gamma\delta}\,. \tag{4.10}$$

For the variation of the Cotton tensor under local Weyl rescalings we have

$$\delta_\sigma \tilde{C}_{\beta\gamma\delta} = \partial_\alpha\sigma\, C^{\alpha}{}_{\beta\gamma\delta}\,. \tag{4.11}$$

The formula of Fefferman and Graham [4] can be written as

$$H = V^{\alpha\beta\gamma\delta\epsilon}V_{\alpha\beta\gamma\delta\epsilon} + 16\,\tilde{C}^{\gamma\delta\varepsilon}\tilde{C}_{\gamma\delta\varepsilon} + 16\,C^{\alpha\gamma\delta\varepsilon}(-\nabla_\alpha\tilde{C}_{\gamma\delta\varepsilon} + K_{\alpha\beta}C^{\beta}{}_{\gamma\delta\varepsilon})\,, \tag{4.12}$$

where

$$V_{\alpha\beta\gamma\delta\varepsilon} \equiv \nabla_\alpha C_{\beta\gamma\delta\varepsilon} + 2(g_{\alpha[\beta}\tilde{C}_{\gamma]\delta\varepsilon} + g_{\alpha[\delta}\tilde{C}_{\varepsilon]\beta\gamma})\,. \tag{4.13}$$

Using (4.10), the invariant (4.12) may alternatively be written as

$$\begin{aligned} H &= \nabla^\alpha C^{\beta\gamma\delta\varepsilon}\nabla_\alpha C_{\beta\gamma\delta\varepsilon} - 4(d-10)\tilde{C}^{\gamma\delta\varepsilon}\tilde{C}_{\gamma\delta\varepsilon} \\ &\quad + 16\,(-C^{\alpha\gamma\delta\varepsilon}\nabla_\alpha\tilde{C}_{\gamma\delta\varepsilon} +\, K_{\alpha\beta}C^{\alpha\gamma\delta\varepsilon}C^{\beta}{}_{\gamma\delta\varepsilon})\,. \end{aligned} \tag{4.14}$$

[2] See the second paper in [1] or in [12] with corrections in [9]

The variation under local Weyl rescalings may be calculated with the aid of

$$\begin{aligned}\delta_\sigma(\nabla^\alpha C^{\beta\gamma\delta\varepsilon}\nabla_\alpha C_{\beta\gamma\delta\varepsilon}) &= 6\sigma\,\nabla^\alpha C^{\beta\gamma\delta\varepsilon}\nabla_\alpha C_{\beta\gamma\delta\varepsilon} \\ &\quad + \partial_\alpha\sigma\left(4\nabla^\alpha C^{\beta\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon} + 8\nabla^\beta C^{\alpha\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon} - 8C^{\alpha\gamma\delta\varepsilon}\nabla^\beta C_{\beta\gamma\delta\varepsilon}\right)\,, \end{aligned} \tag{4.15}$$

and (4.10), which then ensures (4.6).

Independently, Parker and Rosenberg [5] found an equivalent scalar which can be written in the form

$$\Omega = \frac{d-10}{d-2}\left(\nabla^\alpha C^{\beta\gamma\delta\varepsilon}\nabla_\alpha C_{\beta\gamma\delta\varepsilon} - 4(d-2)\tilde{C}^{\gamma\delta\varepsilon}\tilde{C}_{\gamma\delta\varepsilon}\right) + \frac{4}{d-2}\left(\nabla^2 - 4J\right)C^{\beta\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon}\,. \tag{4.16}$$

With standard results for the commutator

$$\begin{aligned} C^{\alpha\gamma\delta\varepsilon}[\nabla_\alpha,\nabla_\beta]C^\beta{}_{\gamma\delta\varepsilon} &= -(d-2)K_{\alpha\beta}C^{\alpha\gamma\delta\varepsilon}C^\beta{}_{\gamma\delta\varepsilon} - J\,C^{\beta\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon} + \Omega_1 + 2\Omega_2 \\ &= -(d-2)C^{\alpha\gamma\delta\varepsilon}\nabla_\alpha\tilde{C}_{\gamma\delta\varepsilon} - \tfrac{1}{2}\,C^{\alpha\gamma\delta\varepsilon}\nabla^2 C_{\alpha\gamma\delta\varepsilon}\,, \end{aligned} \tag{4.17}$$

where in the second line we use $C^{\alpha\gamma\delta\varepsilon}\nabla^\beta\nabla_\alpha C_{\beta\gamma\delta\varepsilon} = \frac{1}{2}C^{\alpha\gamma\delta\varepsilon}\nabla^2 C_{\alpha\gamma\delta\varepsilon} - C^{\alpha\gamma\delta\varepsilon}\nabla_\varepsilon\tilde{C}_{\delta\gamma\alpha}$, from (4.9), and (4.10) for $\nabla^\beta C_{\beta\gamma\delta\varepsilon}$. Using (4.17), with H as in (4.14), we then find

$$H = \Omega + \Omega_1 + 2\Omega_2\,. \tag{4.18}$$

It is perhaps worth noting that the formula (4.16) can also be written in the form

$$\begin{aligned}\Omega &= -\frac{1}{d-2}\Big((d-10)C^{\beta\gamma\delta\varepsilon}\nabla^2 C_{\beta\gamma\delta\varepsilon} + 16J\,C^{\beta\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon}\Big) \\ &\quad - 4(d-10)\,\tilde{C}^{\gamma\delta\varepsilon}\tilde{C}_{\gamma\delta\varepsilon} + \tfrac{1}{2}\nabla^2(C^{\beta\gamma\delta\varepsilon}C_{\beta\gamma\delta\varepsilon})\,. \end{aligned} \tag{4.19}$$

5 Conformal Operator on Weyl Tensor Fields

As a further example of a conformally covariant second order differential operator we now construct a second order differential operator acting on tensor fields, $\mathcal{C}_{\mu\sigma\rho\nu}$, with Weyl symmetry, i.e. they satisfy the symmetry and traceless conditions of the Weyl tensor exhibited in (4.2). We assume that $\mathcal{C}_{\mu\sigma\rho\nu}$ is to be regarded as metric independent but also to transform with local rescalings of the metric $\delta_\sigma g^{\mu\nu} = 2\sigma g^{\mu\nu}$ according to

$$\delta_\sigma \mathcal{C}_{\mu\sigma\rho\nu} \;=\; -\,k\,\sigma\,\mathcal{C}_{\mu\sigma\rho\nu}\,, \tag{5.1}$$

with k an arbitrary number.

In order to find such a conformally covariant covariant differential operator, we construct a reparametrisation action of second order in $\mathcal{C}$ and then determine the restrictions necessary to ensure local scale invariance (a similar procedure was followed in a recent

paper by O'Raifeartaigh, Sachs and Wiesendanger [13]). We therefore first consider the d-dimensional action with two derivatives, which by using the symmetry properties of $\mathcal{C}_{\mu\sigma\rho\nu}$, can be reduced to

$$S_0[g,\mathcal{C}] \,=\, \tfrac{1}{2}\int \mathrm{d}^d x\,\sqrt{g}\,\Big[a\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} \,+\, b\,\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}\nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu}\Big]\,. \tag{5.2}$$

The two terms in (5.2) with coefficients a and b are the only possible independent scalars involving $\mathcal{C}_{\mu\sigma\rho\nu}$. In the following we determine the values for the two coefficients a and b necessary for a conformally covariant second order differential operator acting on $\mathcal{C}_{\mu\sigma\rho\nu}$ by requiring $\delta_\sigma S^C[g,\mathcal{C}] = 0$, where $S^C[g,\mathcal{C}]$ contains further curvature dependent terms in addition to $S_0[g,\mathcal{C}]$. It is easy to see that for the action S_0 to be invariant for constant σ then we must have

$$k = -\tfrac{1}{2}(d-10)\,. \tag{5.3}$$

To calculate the variation for general σ we make use of

$$\begin{aligned}\delta_\sigma\left(\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu}\right) \;&=\; (10-2k)\,\sigma\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} + 2(4-k)\partial_\alpha\sigma\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}\\ &\quad + 8\partial_\mu\sigma\,\nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\alpha\sigma\rho\nu} \,-\, 8\partial_\alpha\sigma\,\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^\alpha{}_{\sigma\rho\nu}\,, \qquad (5.4)\\ \delta_\sigma \nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu} \;&=\; (2-k)\sigma\,\nabla^\alpha \mathcal{C}_{\alpha\sigma\rho\nu} \,+\, (5-k-d)\partial_\alpha\sigma\,\mathcal{C}^\alpha{}_{\sigma\rho\nu}\,. \qquad (5.5)\end{aligned}$$

Using these results for (5.2) with (5.1,5.3) we obtain

$$\begin{aligned}\delta_\sigma S_0[g,\mathcal{C}] \;=\; \tfrac{1}{2}\int \mathrm{d}^d x\sqrt{g}\,\Big[\,&a\Big(\tfrac{1}{2}(d-2)\partial_\alpha\sigma\nabla^\alpha(\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}) - 16\partial_\alpha\sigma\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}\,\mathcal{C}^\alpha{}_{\sigma\rho\nu}\\ &-8\,\nabla_\alpha\partial_\mu\sigma\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^\alpha{}_{\sigma\rho\nu}\Big) - bd\,\partial_\alpha\sigma\nabla_\mu \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^\alpha{}_{\sigma\rho\nu}\,\Big]\,.\end{aligned} \tag{5.6}$$

Hence if we choose

$$16\,a = -d\,b\,, \tag{5.7}$$

the terms involving single derivatives of σ in $\delta_\sigma S_0$ cancel and now

$$\delta_\sigma S_0[g,\mathcal{C}] = -\tfrac{1}{2}a\int \mathrm{d}^d x\sqrt{g}\,\Big[\tfrac{1}{2}(d-2)\nabla^2\sigma\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} \,+ 8\nabla_\alpha\partial_\mu\sigma\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^\alpha{}_{\sigma\rho\nu}\,\Big]\,. \tag{5.8}$$

These terms may be cancelled by the conformal variation of the additional action

$$S_1[g,\mathcal{C}] = \tfrac{1}{2}a\int \mathrm{d}^d x\sqrt{g}\,\Big[\tfrac{1}{2}(d-2)J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} + 8K_{\mu\alpha}\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^\alpha{}_{\sigma\rho\nu}\Big] \tag{5.9}$$

involving the curvature dependent terms J and $K_{\mu\nu}$ defined in (3.3) and (3.4). Hence, assuming now $a = d/16, b = -1$, we obtain a conformally invariant action

$$\begin{aligned}S^C[g,\mathcal{C}] \;&=\; S_0[g,\mathcal{C}] + S_1[g,\mathcal{C}]\\ &=\; \tfrac{1}{2}\int \mathrm{d}^d x\sqrt{g}\,\mathcal{C}^{\mu\sigma\rho\nu}(\Delta^C\mathcal{C})_{\mu\sigma\rho\nu}\,,\end{aligned} \tag{5.10}$$

which defines the conformally covariant differential operator Δ^C where explicitly

$$
\begin{aligned}
(\Delta^C \mathcal{C})_{\mu\sigma\rho\nu} \;=\;& -\tfrac{1}{16} d \Big(\nabla^2 - \tfrac{1}{2}(d-2)J\Big)\mathcal{C}_{\mu\sigma\rho\nu} \\
& + \mathcal{E}^C{}_{\mu\sigma\rho\nu,}{}^{\mu'\sigma'\rho'\nu'} \Big(\nabla_{\mu'}\nabla^\alpha + \tfrac{1}{2} d\, K_{\mu'}{}^\alpha\Big)\mathcal{C}_{\alpha\sigma'\rho'\nu'} \,, \qquad (5.11)
\end{aligned}
$$

with $\mathcal{E}^C$ as defined in appendix A.1. As a consequence of $\delta_\sigma S^C[g,\mathcal{C}] = 0$, as well as (5.1) and (5.3), it must satisfy

$$
\delta_\sigma \Delta^C = \tfrac{1}{2}(d-6)\,\sigma\Delta^C - \tfrac{1}{2}(d-10)\,\Delta^C\sigma \,. \qquad (5.12)
$$

For generality we could also add $S_2[g,\mathcal{C}]$ containing the invariants

$$
S_2[g,\mathcal{C}] = \tfrac{1}{2} \int \mathrm{d}^d x \sqrt{g}\, \Big[c\, \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\beta\sigma\alpha} C^\alpha{}_\rho{}^\beta{}_\mu + e\, \mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\rho\nu\alpha\beta} C_{\mu\sigma}{}^{\alpha\beta}\Big] \,, \qquad (5.13)
$$

where $C_{\mu\kappa\lambda\nu}$ is the standard Weyl tensor defined in (1.6). Letting $\tilde{S}^C[g,\mathcal{C}] = S^C[g,\mathcal{C}] + S_2[g,\mathcal{C}]$ we therefore obtain a two parameter conformal differential operator $\tilde{\Delta}^C$ depending on c and e,

$$
\begin{aligned}
(\tilde{\Delta}^C \mathcal{C})_{\mu\sigma\rho\nu} \;=\;& (\Delta^C \mathcal{C})_{\mu\sigma\rho\nu} \\
& + \mathcal{E}^C{}_{\mu\sigma\rho\nu,}{}^{\mu'\sigma'\rho'\nu'} \Big(c\, C^\alpha{}_{\rho'}{}^\beta{}_{\mu'}\mathcal{C}_{\nu'\beta\sigma'\alpha} + e\, C_{\mu'\sigma'}{}^{\alpha\beta}\mathcal{C}_{\rho'\nu'\alpha\beta}\Big) \,. \qquad (5.14)
\end{aligned}
$$

The corresponding Green function is defined by

$$
\sqrt{g(x)}\,(\Delta_x{}^C G^C)_{\mu\sigma\rho\nu,}{}^{\alpha\gamma\delta\beta}(x,y) = \mathcal{E}^C{}_{\mu\sigma\rho\nu,}{}^{\alpha\gamma\delta\beta}\, \delta^d(x-y) \,. \qquad (5.15)
$$

On flat space this may be calculated in a similar fashion as previously by inverting the Fourier transform of Δ^C. From the results in appendix A.3

$$
\mathring{G}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) = \frac{1}{S_d}\, \frac{16}{(d-4)(d-6)}\, \frac{1}{x^{d-2}}\, \mathcal{I}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) \,, \qquad (5.16)
$$

where the relevant form for the inversion tensor is

$$
\mathcal{I}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) = I_{\mu\epsilon}(x) I_{\sigma\kappa}(x) I_{\rho\lambda}(x) I_{\nu\eta}(x) \mathcal{E}^C{}_{\epsilon\kappa\lambda\eta,\alpha\gamma\delta\beta} \,. \qquad (5.17)
$$

This is clearly in accord with the general expression expected by conformal invariance given by (2.21) which thus provides a consistency check on the derivation of the conformal operator Δ^C. Obviously from (5.16) Δ^C is not invertible $d = 4$ or $d = 6$.

The lack of an inverse for $d = 4, 6$ is true more generally. To demonstrate this we first note that Δ^C essentially vanishes identically if $d = 4$. For $d = 4$ from the vanishing of totally antisymmetric five index tensors we have

$$
\begin{aligned}
0 \;=\;& 5\, \nabla^\alpha \mathcal{C}^{\mu\sigma}{}_{[\rho\nu} \nabla_\alpha \mathcal{C}_{\mu\sigma]}{}^{\rho\nu} \\
=\;& \nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} - 2\, \nabla^\alpha \mathcal{C}_\alpha{}^{\sigma\rho\nu} \nabla_\mu \mathcal{C}^\mu{}_{\sigma\rho\nu} - 2\, \nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\mu \mathcal{C}_{\alpha\sigma\rho\nu} \\
\to\;& \nabla^\alpha \mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha \mathcal{C}_{\mu\sigma\rho\nu} - 4 \nabla_\alpha \mathcal{C}^{\alpha\sigma\rho\nu}\nabla^\mu \mathcal{C}_{\mu\sigma\rho\nu} + 2\, \mathcal{C}^{\mu\sigma\rho\nu}[\nabla^\alpha, \nabla_\mu]\mathcal{C}_{\alpha\sigma\rho\nu} \,, \qquad (5.18)
\end{aligned}
$$

after discarding a total derivative and where for general d

$$\begin{aligned}\mathcal{C}^{\mu\sigma\rho\nu}[\nabla^\alpha,\nabla_\mu]\mathcal{C}_{\alpha\sigma\rho\nu} &= (d-2)\,K_{\mu\alpha}\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^{\alpha}{}_{\sigma\rho\nu} - J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu} \\ &\quad - C^{\alpha\beta}{}_{\mu\sigma}\mathcal{C}^{\mu\sigma}{}_{\rho\nu}\mathcal{C}^{\rho\nu}{}_{\alpha\beta} - 2\,C^{\alpha}{}_{\rho}{}^{\beta}{}_{\mu}\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\beta\sigma\alpha}\,. \end{aligned}\tag{5.19}$$

Similarly we have, using $K_\alpha{}^\alpha = J$,

$$0 = 5\,\mathcal{C}^{\mu\sigma}{}_{[\rho\nu}K_\alpha{}^\alpha\mathcal{C}_{\mu\sigma]}{}^{\rho\nu} = -4K_{\mu\alpha}\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^{\alpha}{}_{\sigma\rho\nu} + J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}\,. \tag{5.20}$$

In order to be able to apply these identities we write the action S^C given in (5.10) in four dimensions as

$$\begin{aligned} S^C[g,\mathcal{C}]\Big|_{d=4} &= \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{g}\Big[\tfrac{1}{4}\,\nabla^\alpha\mathcal{C}^{\mu\sigma\rho\nu}\nabla_\alpha\mathcal{C}_{\mu\sigma\rho\nu} - \nabla_\mu\mathcal{C}^{\mu\sigma\rho\nu}\nabla^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} \\ &\qquad + 2\,K_{\mu\alpha}\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^{\alpha}{}_{\sigma\rho\nu} + \tfrac{1}{2}\,J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}\Big] \\ &= \tfrac{1}{2}\int \mathrm{d}^4x\sqrt{g}\Big[\tfrac{1}{4}\,C^{\alpha\beta}{}_{\mu\sigma}\mathcal{C}^{\mu\sigma}{}_{\rho\nu}\mathcal{C}^{\rho\nu}{}_{\alpha\beta} + \tfrac{1}{2}\,C^{\alpha}{}_{\rho}{}^{\beta}{}_{\mu}\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\nu\beta\sigma\alpha}\Big]\,. \end{aligned}\tag{5.21}$$

by virtue of (5.18), (5.19) and (5.20). For $d=4$ the two $\mathrm{O}(C\mathcal{C}^2)$ terms are not independent, just as in (4.5), so in order to obtain $\tilde{S}^C[g,\mathcal{C}]=0$ and hence $(\tilde{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu}=0$ it is sufficient to take

$$c + 4\,e = -\tfrac{3}{2}\,. \tag{5.22}$$

In six dimensions the lack of an inverse follows since the Weyl tensor itself is a zero mode, i.e.

$$(\tilde{\Delta}^C C)_{\mu\sigma\rho\nu} = 0 \quad \text{ if } \quad c = -\tfrac{3}{2}\,, \quad e = -\tfrac{3}{4}\,. \tag{5.23}$$

Applying Δ^C to the Weyl tensor is consistent for $d=6$ since the transformation of $C_{\mu\sigma\rho\nu}$ in (4.1) matches that for $\mathcal{C}_{\mu\sigma\rho\nu}$ in (5.1) as then $k=2$ from (5.3), To verify (5.23) for this case we now use (4.17) with (4.10) for $d=6$ to give

$$S^C[g,C]\Big|_{d=6} = -\tfrac{3}{8}\int \mathrm{d}^6x\sqrt{g}\,(\Omega_1 + 2\Omega_2)\,. \tag{5.24}$$

This may be cancelled by $S_2[g,C]$, defined in (5.13), for the above choices of the parameters c, e, and as $\tilde{S}[g,C]=0$ (5.23) must hold.

As in (3.19) we may also define

$$2\frac{\delta}{\delta g^{\alpha\beta}}S^C(g,\mathcal{C}) = \sqrt{g}\,T^C{}_{\alpha\beta}\,, \quad \mathring{T}^C{}_{\alpha\beta} = T^C{}_{\alpha\beta}\Big|_{g=\delta}\,. \tag{5.25}$$

The calculation of the flat space expression for $\mathring{T}^C{}_{\alpha\beta}$ is again straightforward although tedious from (5.11) and (5.2), with the assumed values of a, b, and also (5.9). The explicit result is given in appendix A.4. From this we may obtain

$$\begin{aligned} \mathring{T}^C{}_{\alpha\alpha} &= -\tfrac{1}{2}(d-10)\,\mathcal{C}_{\mu\sigma\rho\nu}\,(\mathring{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu}\,, \\ \partial_\alpha\mathring{T}^C{}_{\alpha\beta} &= -(\mathring{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu}\,\partial_\beta\mathcal{C}_{\mu\sigma\rho\nu} + 4\,\partial_\alpha(\mathcal{C}_{\beta\sigma\rho\nu}\,(\mathring{\Delta}^C\mathcal{C})_{\alpha\sigma\rho\nu})\,, \end{aligned}\tag{5.26}$$

with $\mathring{\Delta}^C$ the flat space restriction of Δ^C.

Using (2.24) we may find the variation of the Green function defined in (5.15), in analogy to the k-form case (3.27). As before, assuming the result to be of the form (2.29), which in this case gives

$$\begin{aligned} &\mathring{G}^{C\prime}{}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\beta}(x,y;z) \\ &\quad = -\mathring{G}^{C}{}_{\mu\kappa\lambda\nu,\mu'\kappa'\lambda'\nu'}(x-z)\mathring{G}^{C}{}_{\sigma\varepsilon\eta\rho,\sigma'\varepsilon'\eta'\rho'}(y-z)\,P_{\mu'\kappa'\lambda'\nu',\sigma'\varepsilon'\eta'\rho',\alpha\beta}(Z)\,, \end{aligned} \tag{5.27}$$

with $\mathring{G}^C$ given by (5.16), simplifies the calculation considerably. It is sufficient to determine the most singular terms as $z \to y$. Using (5.25) with the flat space result (A.20) of appendix A.4 gives a relatively lengthy expression for $P_{\mu'\kappa'\lambda'\nu',\sigma'\varepsilon'\eta'\rho',\alpha\beta}(Z)$ which may be found in appendix A.5. An important consistency check is that the conservation equations (2.35) are satisfied.

6 Fourth Order Conformal Operators

So far we have only dealt with second order operators. In this section we discuss some results for fourth order operators.

A fourth order d-dimensional conformal differential operator acting on scalars was found by Paneitz [14] and almost simultaneously in $d=4$ by Riegert [8] and later independently by Eastwood and Singer [15]. This has the form

$$\Delta_4 = \nabla^2\nabla^2 + \nabla_\mu(4K^{\mu\nu} - (d-2)g^{\mu\nu}J)\partial_\nu + \tfrac{1}{2}(d-4)\,M\,, \tag{6.1}$$

where J and K are given by (3.3) and (3.4) and

$$M = -\nabla^2 J + \tfrac{1}{2}dJ^2 - 2K^{\mu\nu}K_{\mu\nu}\ . \tag{6.2}$$

This operator is conformally covariant in the sense that

$$\delta_\sigma \Delta_4 = \tfrac{1}{2}(d+4)\,\sigma\Delta_4 - \tfrac{1}{2}(d-4)\,\Delta_4\sigma\,. \tag{6.3}$$

Subsequently Δ_4 was generalised to differential forms by Branson and also by Wünsch [3, 6]. This operator has the form, with the same notation as in section 3,

$$\Delta_4^{(k)} = \mathcal{D}_4^{(k)} + \text{curvature dependent terms}\,, \quad \mathcal{D}_4^{(k)} = (\gamma+2)\,\delta\mathrm{d}\delta\mathrm{d} + (\gamma-2)\,\mathrm{d}\delta\mathrm{d}\delta\,, \tag{6.4}$$

which satisfies

$$\delta_\sigma \Delta_4^{(k)} = (\gamma+2)\,\sigma\Delta_4^{(k)} - (\gamma-2)\,\Delta_4^{(k)}\sigma\,. \tag{6.5}$$

Acting on functions it reduces to the operator in (6.1)

$$\Delta_4^{(0)} = \tfrac{1}{2}(d+4)\Delta_4\,. \tag{6.6}$$

The Green function $G_4^{(k)}(x,y)$ in for $\Delta_4^{(k)}$ is defined similarly to (3.21). On flat space, this Green function may be calculated by the same technique as used for the second order operators in appendices A.2 and A.3. First we note that in $\mathcal{D}_4^{(k)}$ in (6.4) may be rewritten as

$$\mathcal{D}_4^{(k)} = ((\gamma+2)(\delta \mathrm{d} + \mathrm{d}\delta) - 4\, \mathrm{d}\delta)(\delta \mathrm{d} + \mathrm{d}\delta)\,, \tag{6.7}$$

so that when acting on k-forms, the flat space restriction of the operator $\Delta_4^{(k)}$ is easily seen to be

$$\Big(\mathring{\Delta}_4^{(k)} \mathcal{A}\Big)_{\mu_1\cdots\mu_k}\Big|_{g=\delta} = (\gamma+2)\, \partial^2\partial^2 \mathcal{A}_{\mu_1\cdots\mu_k} - 4k\, \partial^2 \partial_\lambda \partial_{[\mu_1} \mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]} \tag{6.8}$$

By inverting the Fourier transform of this expression and transforming back to position space with the help of (A.6,A.7) and (A.15), we obtain for the Green function on flat space

$$\mathring{G}_4^{(k)}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = \frac{\Gamma(\frac{1}{2}d-1)}{16\pi^{d/2}(\gamma-2)(\gamma+2)}\, \mathcal{I}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\, \frac{1}{x^{d-4}}\,. \tag{6.9}$$

Just as in (3.23) this involves the inversion tensor for k-forms as required by conformal invariance.

For further discussion for simplicity we restrict our attention to the Green function $G_4(x,y)$ for the scalar operator defined in (6.1). From (6.3) this behaves under rescalings of the metric according to

$$\delta_\sigma G_4(x,y) = \tfrac{1}{2}(d-4)(\sigma(x)+\sigma(y))G_4(x,y)\,. \tag{6.10}$$

On restriction to flat space it is easy to see either directly or from (6.9) for $k=0$ using (6.6),

$$\mathring{G}_4(x) = \frac{\Gamma(\frac{1}{2}d-2)}{16\pi^{d/2}}\,\frac{1}{x^{d-4}} = \frac{1}{2(d-2)(d-4)}\,\frac{1}{S_d}\,\frac{1}{x^{d-4}}\,. \tag{6.11}$$

The calculation of the variation of G_4 with respect to the metric is straightforward by taking account of the metric dependence of Δ_4, noting that $\nabla^2 = (\sqrt{g})^{-1}\partial_\mu \sqrt{g} g^{\mu\nu}\partial_\nu$, and gives

$$\begin{aligned}
\mathring{G}'_{4,\alpha\beta}(x,y;z) \;=\; -\frac{1}{d-1}\Big[& -\tfrac{1}{4}(d-4)\,(X\,\partial_\alpha\partial_\beta\partial^2 Y + \partial_\alpha\partial_\beta\partial^2 X\, Y) \\
& + \partial_\alpha\partial_\beta\partial_\rho X\,\partial_\rho Y + \partial_\rho X\,\partial_\alpha\partial_\beta\partial_\rho Y \\
& - \tfrac{1}{2}(d+2)\,(\partial_{(\alpha} X \partial_{\beta)}\,\partial^2 Y + \partial^2\partial_{(\alpha} X\,\partial_{\beta)} Y) \\
& + \tfrac{1}{2}\,\delta_{\alpha\beta}(\partial^2\partial_\rho X\,\partial_\rho Y + \partial_\rho X\,\partial^2\partial_\rho Y) \\
& + \frac{d(d+2)}{4(d-2)}\Big(\partial^2 X\,\partial_\alpha\partial_\beta Y + \partial_\alpha\partial_\beta X\,\partial^2 Y - \frac{2}{d}\,\delta_{\alpha\beta}\partial^2 X\,\partial^2 Y\Big) \\
& - \frac{2d}{d-2}\Big(\partial_\rho\partial_{(\alpha} X\,\partial_{\beta)}\partial_\rho Y - \frac{1}{d}\,\delta_{\alpha\beta}\partial_\sigma\partial_\rho X\,\partial_\sigma\partial_\rho Y\Big)\Big]\,,
\end{aligned} \tag{6.12}$$

where we employ the abbreviations

$$X(z) \equiv \mathring{G}_4(z-x)\,, \quad Y(z) \equiv \mathring{G}_4(z-y)\,. \tag{6.13}$$

The explicit calculation of the derivatives yields an expression in agreement with the general result (2.29),

$$\mathring{G}'_{4,\alpha\beta}(x,y;z) = -\frac{d(d-2)(d-4)^2}{2(d-1)}\,\mathring{G}_4(x-z)\mathring{G}_4(y-z)\,\Big(Z_\alpha Z_\beta - \frac{1}{d}\delta_{\alpha\beta}Z^2\Big)Z^2\,. \tag{6.14}$$

This satisfies the conservation condition (2.35) since it reduces just to (3.31) again.

7 Discussion

A crucial motivation for this work was to use the Green functions of conformally covariant differential operators to construct expressions for the effective action depending on the metric which reproduce exactly the required results for scale anomalies. In four dimensions the operator Δ_4 reduces to the operator introduced by Riegert

$$\Delta^{\rm R} \equiv \Delta_4\Big|_{d=4} = \nabla^2\nabla^2 + 2\nabla_\mu(R^{\mu\nu} - \tfrac{1}{3}g^{\mu\nu}R)\partial_\nu\,, \tag{7.1}$$

which gives $\delta_\sigma(\sqrt{g}\Delta^{\rm R}) = 0$. Its Green function $G^{\rm R}(x,y) = G_4(x,y)|_{d=4}$, which is therefore invariant under local rescalings of the metric $\delta_\sigma G^{\rm R}(x,y) = 0$, was used by Riegert to construct possible forms for the effective action. To demonstrate this we may note that

$$\mathcal{G} \equiv \sqrt{g}(G - \tfrac{2}{3}\nabla^2 R)\,, \tag{7.2}$$

with G the Gauß-Bonnet term defined in (1.5), has the conformal variation

$$\delta_\sigma\mathcal{G} = -4\sqrt{g}\Delta^{\rm R}\sigma\,, \tag{7.3}$$

and hence

$$\delta_\sigma\Sigma = \sigma\,, \qquad \Sigma(x) = -\tfrac{1}{4}\int {\rm d}^4y\, G^{\rm R}(x,y)\mathcal{G}(y)\,. \tag{7.4}$$

If we generalise (1.3) slightly to

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle = -\mathcal{F} - \beta_b\, G \quad \text{for} \quad \delta_\sigma\mathcal{F} = 4\mathcal{F}\,, \tag{7.5}$$

then a four dimensional non-local action, analogous to the two dimensional Polyakov action (1.8), which generates the trace anomaly under local rescalings of the metric was constructed by Riegert [8] of the form,

$$\begin{aligned} W_{\rm Riegert}[g] \;=\;& \int {\rm d}^4x\sqrt{g}\,\mathcal{F}(x)\Sigma(x) \\ & - \tfrac{1}{8}\beta_b\int {\rm d}^4x{\rm d}^4y\,\mathcal{G}(x)\,G^{\rm R}(x,y)\,\mathcal{G}(y) + \tfrac{1}{18}\beta_b\int {\rm d}^4x\sqrt{g}\,R^2(x)\,, \end{aligned} \tag{7.6}$$

where, since $\delta_\sigma(\sqrt{g}R^2) = 12\sqrt{g}R\nabla^2\sigma$, the last term cancels the $\nabla^2 R$ term in $\mathcal{G}$.

Just as in two dimensions we may obtain correlation functions involving the energy momentum tensor by functionally differentiating the action and then restricting to flat space. However, as shown in [7], this does not give rise to conformally invariant results for the flat space correlation functions of the energy momentum tensor. This appears to be connected with the fact that for the operator Δ_4 $d = 4$ is a critical dimension. For the second order operator Δ_2 defined in (1.12) the critical dimension is $d = 2$, as is revealed in (3.29). From (3.29) and (3.30) we may find a finite limit for the variation,

$$\lim_{d\to 2} \mathring{G}'_{2,\alpha\beta}(x,y;z) = \frac{1}{2(2\pi)^2}\Big(Z_\alpha Z_\beta - \tfrac{1}{2}\delta_{\alpha\beta}Z^2\Big)\,. \tag{7.7}$$

Using a complex basis and the explicit form for Z_α in (2.30) we then get

$$\begin{aligned}\mathring{G}'_{2,zz}(x_1,x_2;x_3)\Big|_{d\to 2} &= \frac{1}{2(4\pi)^2}\,\frac{(z_1-z_2)^2}{(z_1-z_3)^2(z_2-z_3)^2} \\ &= \frac{1}{2(4\pi)^2}\Big(\frac{1}{(z_1-z_3)^2}+\frac{1}{(z_2-z_3)^2}-\frac{2}{(z_1-z_3)(z_2-z_3)}\Big)\,.\end{aligned} \tag{7.8}$$

It is interesting to compare this conformally covariant limit with the result shown in (1.18) which was obtained directly in two dimensions and is not in accord with conformal invariance (although (1.18) and (7.8) agree on further differentiation with respect to both z_1 and z_2).

A similar picture emerges in relation to the fourth order operators Δ_4, defined for arbitrary d, and the Riegert operator Δ^R in four dimensions. The flat space limit of G^R is also logarithmic since from (6.11) we may take

$$\mathring{G}_4(x) = -\frac{1}{16\pi^2}\,\ln \mu^2 x^2\,. \tag{7.9}$$

Following (7.7) and using (6.14) we may obtain

$$\lim_{d\to 4} \mathring{G}'_{4,\alpha\beta}(x,y;z) = -\frac{1}{3(4\pi^2)^2}\Big(Z_\alpha Z_\beta - \tfrac{1}{4}\delta_{\alpha\beta}Z^2\Big)Z^2\,. \tag{7.10}$$

However the variation of the Riegert Green function gives

$$\mathring{G}^{R\prime}{}_{\alpha\beta}(x,y;z) = G'_{4,\alpha\beta}(x,y;z)\Big|_{d\to 4} - \tfrac{1}{12}\lim_{d\to 4}\Big((d-4)\,(X\,\partial_\alpha\partial_\beta\partial^2 Y + \partial_\alpha\partial_\beta\partial^2 X\,Y)\Big)\,, \tag{7.11}$$

with X, Y as in (6.13,6.11). The difference between taking the limit $d\to 4$ before or after the variation is then

$$\begin{aligned}&\mathring{G}^{R\prime}{}_{\alpha\beta}(x,y;z) - \mathring{G}'_{4,\alpha\beta}(x,y;z)\Big|_{d\to 4} \\ &= \frac{1}{48\pi^4}\Big[\frac{(y-z)_\alpha(y-z)_\beta}{(y-z)^6}+\frac{(x-z)_\alpha(x-z)_\beta}{(x-z)^6}-\tfrac{1}{4}\delta_{\alpha\beta}\Big(\frac{1}{(y-z)^4}+\frac{1}{(x-z)^4}\Big)\Big]\,.\end{aligned} \tag{7.12}$$

The result for taking the limit $d \to 4$ after the variation given in (7.10) is explicitly conformally invariant but correspondingly the expression given by (7.12) for the variation of the Riegert Green function shows that the additional terms violate the naively expected conformal invariance. The additional terms in $G^{R\prime}$ ensure that is less singular as $x, y \to z$ and they disappear on taking derivatives both with respect to x and y. The results obtained in this paper therefore show a very strong parellelism between the properties of the second order operator Δ_2, with critical dimension 2, and the fourth order operator Δ_4, with critical dimension 4.

8 Conclusion

In conclusion we may perhaps reiterate that, in addition to the results for the fourth order conformal operator discussed in the previous section, we have shown that the Green functions of two second order conformal operators lead to conformally invariant expressions on flat space when varied with respect to the metric. For the second order differential operator acting on k-forms this expression is given by (3.27), while for the operator on Weyl tensor fields it is given by (5.27). These results may be useful for constructing a four-dimensional non-local effective action which parallels the two-dimensional Polyakov action in at least the following respects: Like the Polyakov action, this action should generate the conformal anomaly exactly under Weyl transformations. Moreover, when varying this action with respect to the metric, it should also be possible to obtain conformal expressions for the two and three point functions involving the energy momentum tensor.

Acknowledgements

I am very grateful to Hugh Osborn for many useful discussions.

Note added. It was pointed out by T. Branson that according to a classification scheme for conformal operators developed in [16], the conformal operator on Weyl tensor fields discussed in section 5 may be written in the form $\tilde{\Delta}^C = \mathcal{D}^{\wedge *}\mathcal{D}$, where $\mathcal{D}$ is a first order differential operator which is conformally covariant in six dimensions. In agreement

with these results it is possible to construct such an operator explicitly. It is given by

$$(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} = 3\,\mathcal{C}_{\mu\sigma[\rho\nu;\alpha]} - \frac{3}{d-3}\Big(\nabla^\lambda \mathcal{C}_{\lambda\sigma[\rho\nu}g_{\alpha]\mu} - \nabla^\lambda \mathcal{C}_{\lambda\mu[\rho\nu}g_{\alpha]\sigma}\Big)\,, \tag{E.1}$$

such that $g^{\mu\alpha}(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} = 0$ and

$$\begin{aligned} \delta_\sigma(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} \;=\;& \tfrac{1}{2}(d-10)\,\sigma(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} \\ &+ \tfrac{3}{2}(d-6)\Big\{\mathcal{C}_{\mu\sigma[\rho\nu}\partial_{\alpha]}\sigma - \frac{1}{d-3}\,\partial^\lambda\sigma\Big(\mathcal{C}_{\lambda\sigma[\rho\nu}g_{\alpha]\mu} - \mathcal{C}_{\lambda\mu[\rho\nu}g_{\alpha]\sigma}\Big)\Big\}\,. \end{aligned} \tag{E.2}$$

Thus $\mathcal{D}$ is conformally covariant when $d = 6$. Using (E.1) we may write the action $\tilde{S}^C[g,\mathcal{C}]$ involving the operator $\tilde{\Delta}^C$ defined in (5.14), letting $c = -\frac{1}{4}d$, $e = -\frac{1}{8}d$, in the form

$$\begin{aligned} \tilde{S}^C[g,\mathcal{C}] \;=\;& \frac{d}{16}\int \mathrm{d}^d x\sqrt{g}\,\Big\{\frac{1}{6}\,(\mathcal{D}\mathcal{C})^{\mu\sigma\rho\nu\alpha}(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} + \frac{(d-4)(d-6)}{d(d-3)}\nabla_\mu\mathcal{C}^{\mu\sigma\rho\nu}\nabla^\alpha\mathcal{C}_{\alpha\sigma\rho\nu} \\ &- (d-6)\Big(K_{\mu\alpha}\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}^{\alpha}{}_{\sigma\rho\nu} - \tfrac{1}{4}J\,\mathcal{C}^{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu}\Big)\Big\}\,. \end{aligned} \tag{E.3}$$

When $d = 6$ we may use the identities of section 4 to show that $(\mathcal{D}C)_{\mu\sigma\rho\nu\alpha} = 0$ when acting on the genuine metric dependent Weyl tensor C, so that $\tilde{S}^C[g,C] = 0$. When $d = 4$ $(\mathcal{D}\mathcal{C})^{\mu\sigma\rho\nu\alpha}(\mathcal{D}\mathcal{C})_{\mu\sigma\rho\nu\alpha} = 0$ and $\tilde{S}^C[g,\mathcal{C}] = 0$ for any $\mathcal{C}$.

Furthermore it was brought to my attention that the fourth order operator on scalars constructed by Riegert and also by Paneitz was discussed independently in [17]. Moreover an early discussion of this operator in general dimensions as well as of conformal operators acting on vectors and on second rank tensors may be found in [18].

A Appendix

A.1 Projection Operator onto the Space of Tensors with Weyl Symmetry

The projection operator $\mathcal{E}^C$ has the explicit form

$$\begin{aligned}\mathcal{E}^C{}_{\mu\sigma\rho\nu,}{}^{\alpha\gamma\delta\beta} \;=\;& \tfrac{1}{12}\Big(\delta_\mu{}^\alpha\delta_\nu{}^\beta\delta_\sigma{}^\gamma\delta_\rho{}^\delta + \delta_\mu{}^\delta\delta_\sigma{}^\beta\delta_\rho{}^\alpha\delta_\nu{}^\gamma - \mu\leftrightarrow\sigma, \nu\leftrightarrow\rho\Big)\\ &+\tfrac{1}{24}\Big(\delta_\mu{}^\alpha\delta_\nu{}^\gamma\delta_\rho{}^\delta\delta_\sigma{}^\beta - \mu\leftrightarrow\sigma,\nu\leftrightarrow\rho,\alpha\leftrightarrow\gamma,\beta\leftrightarrow\delta\Big)\\ &-\frac{1}{d-2}\tfrac{1}{8}\Big(g_{\mu\rho}g^{\alpha\delta}\delta_\sigma{}^\gamma\delta_\nu{}^\beta + g_{\mu\rho}g^{\alpha\delta}\delta_\sigma{}^\beta\delta_\nu{}^\gamma - \mu\leftrightarrow\sigma,\nu\leftrightarrow\rho,\alpha\leftrightarrow\gamma,\beta\leftrightarrow\delta\Big)\\ &+\frac{1}{(d-2)(d-1)}\tfrac{1}{2}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\sigma\rho}\right)\Big(g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\beta}g^{\gamma\delta}\Big)\ . \qquad \text{(A.1)}\end{aligned}$$

This projection operator satisfies the symmetries of the Weyl tensor as given by (4.2) in both sets of indices.

A.2 Green Function for the Differential Operator on k-Forms

Here we calculate the flat space Green function of the operator defined in (3.12). On flat space $\mathrm{d}\delta+\delta\mathrm{d}\to-\partial^2$ so from the result in (3.10) this operator reduces to

$$(\mathring{\Delta}^{(k)}\mathcal{A})_{\mu_1\cdots\mu_k} = -(\gamma+1)\partial^2\mathcal{A}_{\mu_1\cdots\mu_k} \;+\; 2k\,\partial_\lambda\partial_{[\mu_1}\mathcal{A}_{|\lambda|\mu_2\cdots\mu_k]} \qquad \text{(A.2)}$$

with $\gamma=\frac{1}{2}(d-2k)$. Its Fourier transform is

$$\begin{aligned}P_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p) \;=\;& (\gamma+1)\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}p^2\\ &-2k\,\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\,\mathcal{E}^A{}_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}p_\varepsilon p_\eta\,, \qquad \text{(A.3)}\end{aligned}$$

where $\mathcal{E}^A$ is the projector on totally antisymmetric tensors of rank k. The inverse of the Fourier transform is defined by

$$P^{-1}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p)\,P_{\nu_1\cdots\nu_k,\lambda_1\cdots\lambda_k}(p) = \mathcal{E}^A{}_{\mu_1\cdots\mu_k,\lambda_1\cdots\lambda_k} \qquad \text{(A.4)}$$

from which we obtain

$$\begin{aligned}P^{-1}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(p) \;=\;& \frac{1}{(\gamma+1)}\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}\frac{1}{p^2}\\ &+\frac{2k}{(\gamma+1)(\gamma-1)}\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\,\mathcal{E}^A{}_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}\frac{p_\varepsilon p_\eta}{p^4}\,. \qquad \text{(A.5)}\end{aligned}$$

When transforming back to d dimensional position space we may use

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-i\,p\cdot x}\frac{1}{(p^2)^\alpha} = \frac{\Gamma(\frac{1}{2}d-\alpha)}{4^\alpha\pi^{\frac{1}{2}d}\Gamma(\alpha)}\,\frac{1}{x^{d-2\alpha}}\,, \qquad \text{(A.6)}$$

which leads to

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-i\,p\cdot x}\,\frac{p_\varepsilon p_\eta}{p^4} = -\frac{\Gamma(\frac{1}{2}d-2)}{16\pi^{\frac{1}{2}d}}\,\partial_\varepsilon\partial_\eta\frac{1}{x^{d-4}}\,. \tag{A.7}$$

With the aid of these relations

$$\begin{aligned}\widetilde{P^{-1}}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) = {}& \frac{\Gamma(\frac{1}{2}d)}{\pi^{\frac{1}{2}d}}\frac{1}{(d-2k-2)(d-2k+2)}\\ & \times\left[\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}\frac{1}{x^{d-2}} - 2k\,\mathcal{E}^A{}_{\mu_1\cdots\mu_k,\varepsilon\lambda_2\cdots\lambda_k}\,\mathcal{E}^A{}_{\eta\lambda_1\cdots\lambda_k,\nu_1\cdots\nu_k}\frac{x_\varepsilon x_\eta}{x^d}\right]\end{aligned} \tag{A.8}$$

and finally the Green function for the operator (A.2) becomes

$$\begin{aligned}\mathring{G}^{(k)}{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x) &= \widetilde{P^{-1}}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\\ &= \frac{\Gamma(\frac{1}{2}d)}{\pi^{\frac{1}{2}d}(d-2k-2)(d-2k+2)}\,\mathcal{I}^A{}_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k}(x)\,\frac{1}{x^{d-2}}\,,\end{aligned} \tag{A.9}$$

where $\mathcal{I}^A$ is the inversion on k-forms which is defined in (3.25). On functions or 0-forms this reduces to

$$\mathring{G}^{(0)}(x) = \frac{\Gamma(\frac{1}{2}d)}{\pi^{\frac{1}{2}d}(d-2)(d+2)}\,\frac{1}{x^{d-2}}\,, \tag{A.10}$$

which up to a factor $2/(d+2)$ is the standard flat space scalar Green function as given in (3.29).

A.3 Green Function for the Differential Operator on Weyl Tensor Fields

Here we calculate the flat space Green function for the operator Δ^C given by (5.11).

The flat space reduction of the operator Δ^C is

$$(\mathring{\Delta}^C\mathcal{C})_{\mu\sigma\rho\nu} = -\tfrac{1}{16}d\,\partial^2\mathcal{C}_{\mu\sigma\rho\nu} + \mathcal{E}^C{}_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}\partial_{\mu'}\partial_\alpha\mathcal{C}_{\alpha\sigma'\rho'\nu'} \tag{A.11}$$

which has the Fourier transform

$$P^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) = \tfrac{1}{16}\,d\,p^2\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} - p_\kappa p_\lambda\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\kappa\sigma'\rho'\nu'}\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\lambda\sigma'\rho'\nu'}\,. \tag{A.12}$$

The inverse of this Fourier transform is defined by

$$P^C{}_{\mu\sigma\rho\nu,\mu'\sigma'\rho'\nu'}(p)P^{C^{-1}}{}_{\mu'\sigma'\rho'\nu',\alpha\gamma\delta\beta}(p) = \mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\,, \tag{A.13}$$

from which we obtain

$$\begin{aligned}P^{C^{-1}}{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p) = {}& \frac{16}{d}\,\frac{1}{p^2}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\\ & + \frac{256}{d(d-4)}\,\frac{p_\varepsilon p_\eta}{p^4}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\\ & + \frac{2048}{d(d-4)(d-6)}\,\frac{p_\varepsilon p_\eta p_\kappa p_\lambda}{p^6}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\,.\end{aligned} \tag{A.14}$$

To transform this expression back to position space, we use the equations (A.6,A.7) as well as

$$\frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ip\cdot x}\,\frac{p_\varepsilon p_\eta p_\kappa p_\lambda}{p^6} = \frac{\Gamma(\frac{1}{2}d-3)}{\pi^{\frac{1}{2}d}2^7}\,\partial_\varepsilon\partial_\eta\partial_\kappa\partial_\lambda\frac{1}{x^{d-6}} \tag{A.15}$$

which gives

$$\begin{aligned}
\widetilde{P^{C}}^{-1}{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) &= \frac{1}{(2\pi)^d}\int \mathrm{d}^d p\, e^{-ip\cdot x}P^{C}{}^{-1}{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(p)\\
&= \frac{4}{d}\,\frac{\Gamma(\frac{1}{2}d-1)}{\pi^{\frac{1}{2}d}}\Big[\frac{1}{x^{d-2}}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}\\
&\quad+\frac{8}{(d-4)}\Big(\frac{1}{x^{d-2}}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}-(d-2)\frac{x_\varepsilon x_\eta}{x^d}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\Big)\\
&\quad+\frac{16}{(d-4)(d-6)}\Big(\tfrac{3}{2}\frac{1}{x^{d-2}}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}-3(d-2)\frac{x_\varepsilon x_\eta}{x^d}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\\
&\qquad+d(d-2)\frac{x_\varepsilon x_\eta x_\kappa x_\lambda}{x^{d+2}}\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\Big)\Big]\,.
\end{aligned} \tag{A.16}$$

Now the inversion on the space of tensors with Weyl symmetry may be written as

$$\begin{aligned}
\mathcal{I}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) &= \mathcal{E}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta} - 8\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\theta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\eta\varphi\theta\omega}\frac{x_\varepsilon x_\eta}{x^2}\\
&\quad+16\,\mathcal{E}^C{}_{\mu\sigma\rho\nu,\varepsilon\varphi\eta\omega}\,\mathcal{E}^C{}_{\alpha\gamma\delta\beta,\kappa\varphi\lambda\omega}\frac{x_\varepsilon x_\eta x_\kappa x_\lambda}{x^4}\,,
\end{aligned} \tag{A.17}$$

so that we obtain for the flat space Green function of Δ^C defined in (5.15)

$$\begin{aligned}
\mathring{G}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x) &= \widetilde{P^{C}}^{-1}{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)\\
&= \frac{8\,\Gamma(\frac{1}{2}d)}{\pi^{\frac{1}{2}d}(d-4)(d-6)}\,\frac{1}{x^{d-2}}\,\mathcal{I}^C{}_{\mu\sigma\rho\nu,\alpha\gamma\delta\beta}(x)\,,
\end{aligned} \tag{A.18}$$

which is in agreement with the form expected from conformal invariance.

A.4 Variation of Conformal Actions

The result for the flat space limit of the metric variation (3.19) of the k-form action (3.16) is explicitly

$$\begin{aligned}
\mathring{T}^{(k)}{}_{\alpha\beta} &= (\gamma+1)\tfrac{1}{k!}\,(\mathrm{d}\mathcal{A})_{\alpha\mu_1\dots\mu_k}\,(\mathrm{d}\mathcal{A})_{\beta\mu_1\dots\mu_k} - (\gamma-1)\tfrac{1}{(k-2)!}\,(\delta\mathcal{A})_{\alpha\mu_1\dots\mu_{k-2}}\,(\delta\mathcal{A})_{\beta\mu_1\dots\mu_{k-2}}\\
&\quad+2(\gamma-1)\tfrac{1}{(k-1)!}\,\mathcal{A}_{(\alpha|\mu_1\dots\mu_{k-1}}\,(\mathrm{d}\delta\mathcal{A})_{\beta)\mu_1\dots\mu_{k-1}}\\
&\quad-\tfrac{1}{2}\delta_{\alpha\beta}\Big((\gamma+1)\,(\mathrm{d}\mathcal{A})\cdot(\mathrm{d}\mathcal{A})-(\gamma-1)\,(\delta\mathcal{A})\cdot(\delta\mathcal{A})+2(\gamma-1)\,\mathcal{A}\cdot(\mathrm{d}\delta\mathcal{A})\Big)\\
&\quad+\frac{(\gamma+1)(\gamma-1)}{d-2}\,\frac{1}{(k-1)!}\,\Big(2\partial_\mu\partial_{(\alpha}(\mathcal{A}_{\beta)\mu_1\dots\mu_{k-1}}\,\mathcal{A}_{\mu\mu_1\dots\mu_{k-1}})\\
&\qquad-\partial^2(\mathcal{A}_{\alpha\mu_1\dots\mu_{k-1}}\mathcal{A}_{\beta\mu_1\dots\mu_{k-1}})-\delta_{\alpha\beta}\partial_\mu\partial_\nu(\mathcal{A}_{\mu\mu_1\dots\mu_{k-1}}\mathcal{A}_{\nu\mu_1\dots\mu_{k-1}})\Big)\\
&\quad-\frac{(\gamma+1)(\gamma-1)}{(d-2)(d-1)}(\tfrac{1}{2}d-1+k)(\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\partial^2)(\mathcal{A}\cdot\mathcal{A})\,.
\end{aligned} \tag{A.19}$$

The corresponding result for the variation (5.25) reduced to flat space can be calculated straightforwardly albeit tediously from the standard form for the dependence of the connection on the metric, although it is also necessary to take account of the implicit dependence through the traceless condition $g^{\mu\nu}\mathcal{C}_{\mu\sigma\rho\nu}=0$. Without attempting to find the simplest form but leaving the contribution of each term in the action separate we find

$$
\begin{aligned}
\mathring{T}^{C}{}_{\alpha\beta} = {}& \frac{d}{16}\Big[\partial_\alpha\mathcal{C}_{\mu\sigma\rho\nu}\,\partial_\beta\mathcal{C}_{\mu\sigma\rho\nu} - 4\,\partial^2\mathcal{C}_{(\alpha|\sigma\rho\nu}\,\mathcal{C}_{\beta)\sigma\rho\nu} - \tfrac{1}{2}\delta_{\alpha\beta}\,\partial_\gamma\mathcal{C}_{\mu\sigma\rho\nu}\,\partial_\gamma\mathcal{C}_{\mu\sigma\rho\nu} \\
&\qquad -4\,\partial_\mu\big(\partial_{(\alpha|}\mathcal{C}_{\mu\sigma\rho\nu}\,\mathcal{C}_{\beta)\sigma\rho\nu} - \partial_{(\alpha}\mathcal{C}_{\beta)\sigma\rho\nu}\,\mathcal{C}_{\mu\sigma\rho\nu}\big)\Big] \\
& - \partial_\mu\mathcal{C}_{\mu(\alpha|\rho\nu}\,\partial_\lambda\mathcal{C}_{\lambda|\beta)\rho\nu} - 2\,\partial_\mu\mathcal{C}_{\mu\sigma\rho(\alpha}\,\partial_\lambda\mathcal{C}_{\lambda\sigma\rho|\beta)} + 2\,\mathcal{C}_{(\alpha|\sigma\rho\nu}\,\partial_{\beta)}\partial_\mu\mathcal{C}_{\mu\sigma\rho\nu} \\
& - \delta_{\alpha\beta}\Big(\tfrac{1}{2}\,\partial_\mu\mathcal{C}_{\mu\sigma\rho\nu}\,\partial_\lambda\mathcal{C}_{\lambda\sigma\rho\nu} + \mathcal{C}_{\lambda\sigma\rho\nu}\,\partial_\lambda\partial_\mu\mathcal{C}_{\mu\sigma\rho\nu}\Big) \\
& + 3\,\partial_\lambda(\mathcal{C}_{\lambda(\alpha|\rho\nu}\,\partial_\mu\mathcal{C}_{\mu|\beta)\rho\nu}) - 2\,\partial_\rho(\mathcal{C}_{\sigma(\alpha\beta)\nu}\,\partial_\mu\mathcal{C}_{\mu\sigma\rho\nu}) - \frac{4}{d-2}\,\mathcal{C}_{\alpha\sigma\beta\nu}\partial_\lambda\partial_\mu\mathcal{C}_{\mu\sigma\lambda\nu} \\
& - \frac{d}{4(d-2)}\Big[2\,\partial_\mu\partial_{(\alpha}(\mathcal{C}_{\beta)\sigma\rho\nu}\,\mathcal{C}_{\mu\sigma\rho\nu}) - \partial^2(\mathcal{C}_{(\alpha|\sigma\rho\nu}\,\mathcal{C}_{\beta)\sigma\rho\nu}) - \delta_{\alpha\beta}\,\partial_\mu\partial_\lambda(\mathcal{C}_{\mu\sigma\rho\nu}\,\mathcal{C}_{\lambda\sigma\rho\nu})\Big] \\
& - \frac{d(d-6)(d+2)}{64(d-2)(d-1)}\,(\partial_\alpha\partial_\beta - \delta_{\alpha\beta}\,\partial^2)(\mathcal{C}_{\mu\sigma\rho\nu}\mathcal{C}_{\mu\sigma\rho\nu})\,. \qquad \text{(A.20)}
\end{aligned}
$$

A.5 Variation of the Weyl Symmetry Green Function

The tensor $P_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\beta}(Z)$ in (5.27) has the form

$$
\begin{aligned}
& P_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho,\alpha\beta}(Z) \\
={}& A\,\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\alpha'\tau\chi\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\beta'\tau\chi\omega}\mathcal{E}^{T}{}_{\alpha'\beta',\alpha\beta}\,\frac{1}{(Z^2)^{\frac{1}{2}d-2}} \\
&+ B\,\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\sigma\varepsilon\eta\rho}\Big(\frac{Z_\alpha Z_\beta}{Z^2} - \frac{1}{d}\delta_{\alpha\beta}\Big)\frac{1}{(Z^2)^{\frac{1}{2}d-2}} \\
&+ C\,\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\alpha'\theta\chi\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\beta'\phi\chi\omega}\mathcal{E}^{T}{}_{\alpha'\beta',\alpha\beta}\,\frac{Z_\theta Z_\phi}{(Z^2)^{\frac{1}{2}d-1}} \\
&+ D\,\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\alpha'\chi\omega\theta}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\beta'\chi\omega\phi}\mathcal{E}^{T}{}_{\alpha'\beta',\alpha\beta}\,\frac{Z_\theta Z_\phi}{(Z^2)^{\frac{1}{2}d-1}} \\
&+ E\,(\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\alpha'\tau\chi\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\theta\tau\chi\omega} \\
&\qquad + \mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\theta\tau\chi\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\alpha'\tau\chi\omega})\mathcal{E}^{T}{}_{\alpha'\beta',\alpha\beta}\,\frac{Z_\theta Z_{\beta'}}{(Z^2)^{\frac{1}{2}d-1}} \\
&+ F\,(\mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\chi\alpha\beta\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\chi\theta\phi\omega} + \mathcal{E}^{C}{}_{\mu\kappa\lambda\nu,\chi\theta\phi\omega}\mathcal{E}^{C}{}_{\sigma\varepsilon\eta\rho,\chi\alpha\beta\omega})\frac{Z_\theta Z_\phi}{(Z^2)^{\frac{1}{2}d-1}}\,, \qquad \text{(A.21)}
\end{aligned}
$$

with $\mathcal{E}^C$ defined in appendix A.1 and with $\mathcal{E}^{T}{}_{\varepsilon\eta,\alpha\beta} = \frac{1}{2}(\delta_{\varepsilon\alpha}\delta_{\eta\beta}+\delta_{\varepsilon\beta}\delta_{\eta\alpha}) - \frac{1}{d}\delta_{\varepsilon\eta}\delta_{\alpha\beta}$. The coefficients are then given by

$$
A = \frac{d^2}{4(d-1)(d-2)}(d-16)
$$

$$
\begin{aligned}
B &= -\frac{d^2}{64(d-1)(d-2)}(d^3-22d^2+76d-40) \\
C &= -\frac{d}{(d-1)(d-2)}(d^3-2d^2+2d-16) \\
D &= \frac{d}{(d-1)(d-2)}(d^3-8d^2+20d-28) \\
E &= \frac{d}{8(d-1)(d-2)}(d^4-7d^3+24d^2-36d+48) \\
F &= \frac{d}{2(d-1)(d-2)}(d^3-8d^2+18d+4)\,.
\end{aligned}
\tag{A.22}
$$

These coefficients satisfy two linear relations - derived in [7] - which follow from conservation,

$$
\mathcal{T}_1 \equiv \tfrac{1}{2}(d-4)(d+4)A - 8B - \tfrac{1}{2}(d+2)C + \tfrac{1}{4}(d-4)D - 2E - \tfrac{3}{2}dF \quad = 0\,, \tag{A.23}
$$

$$
\mathcal{T}_2 \equiv -(d-4)(d+4)A + 16B + dC + \tfrac{1}{4}d(d-4)D - \tfrac{1}{2}d(d-10)F \quad = 0\,. \tag{A.24}
$$

This provides a direct check on the conservation of $\mathring{G}^{C\prime}$.

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